A 4.3

(a) $$Y(z) - \frac{1}{2} z^{-1} Y(z) + \frac{1}{4} z^{-2} Y(z) = X(z)$$

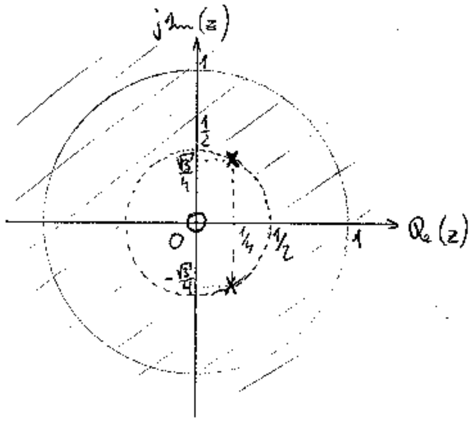
$$Y(z) \cdot \left\{ 1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} \right\} = X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}} = \frac{4z^2}{4z^2 - 2z + 1} = \frac{z^2}{z^2 - \frac{1}{2} z + \frac{1}{4}}$$

(b) NST: $z^2 = 0 \Rightarrow z_{0\,1,2} = 0$

POL: $z^2 - \frac{1}{2} z + \frac{1}{4} = 0$

$$z_{\infty 1,2} = \frac{1}{4} \pm j \frac{\sqrt{3}}{4} = \frac{1}{2} \cdot e^{\pm j \frac{\pi}{3}}$$



(c) $$X(z) = \frac{z}{z - \frac{1}{2}}$$

(i) $$Y(z) = \frac{z^2}{z^2 - \frac{1}{2} z + \frac{1}{4}} \cdot \frac{z}{z - \frac{1}{2}} = \frac{z^3}{z^3 - z^2 + \frac{1}{2} z - \frac{1}{8}} \overset{PD}{=} 1 + \frac{z^2 - \frac{1}{2} z + \frac{1}{8}}{\left(z^2 - \frac{1}{2} z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} = 1 + \frac{A}{z - \frac{1}{2}} + \frac{Bz + C}{\left(z - \frac{1}{4}\right)^2 + \frac{3}{16}}$$

$$A = \lim_{z \to \frac{1}{2}} \frac{z^2 - \frac{1}{2} z + \frac{1}{8}}{z^2 - \frac{1}{2} z + \frac{1}{4}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$

$$z = \frac{1}{4}: \frac{\frac{1}{16}}{\frac{3}{16} \cdot \left(-\frac{1}{4}\right)} = \frac{\frac{1}{2}}{-\frac{1}{4}} + \frac{\frac{B}{4} + C}{\frac{3}{16}}$$

$$-\frac{4}{3} = -2 + \frac{4}{3} B + \frac{16}{3} C$$

$$1 = 2B + 8C$$

$$z = 0: \frac{\frac{1}{8}}{-\frac{1}{8}} = \frac{\frac{1}{2}}{-\frac{1}{2}} + \frac{C}{\frac{1}{4}}$$

$\Rightarrow C = 0$

$\Rightarrow B = \frac{1}{2}$

$$Y(z) = 1 + \frac{1}{2} z^{-1} \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{2} z}{z^2 - \frac{1}{2} z + \frac{1}{4}}$$

$$\frac{\frac{1}{2} z}{z^2 - \frac{1}{2} z + \frac{1}{4}} = k \cdot \frac{\rho z \sin(\alpha)}{z^2 - 2\rho z \cos(\alpha) + \rho^2} \Rightarrow$$

$\rho = \frac{1}{2}$

$\cos(\alpha) = \frac{1}{2}$

$\Rightarrow \sin(\alpha) = \frac{\sqrt{3}}{2}$

$\Rightarrow k = \frac{2}{\sqrt{3}}$

A 4.3

(c)

$$Y(z) = 1 + \frac{1}{2} z^{-1} \frac{z}{z-\frac{1}{2}} + \frac{2}{\sqrt{3}} \cdot \frac{\frac{1}{2} \cdot z \cdot \sin\left(\frac{\pi}{3}\right)}{z^2 - 2 \cdot \frac{1}{2} \cdot z \cdot \cos\left(\frac{\pi}{3}\right) + \left(\frac{1}{2}\right)^2}$$

$$y[n] = \delta[n] + \frac{1}{2} \left(\frac{1}{2}\right)^{n-1} \sigma[n-1] + \frac{2}{\sqrt{3}} \cdot \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3} n\right) \sigma[n] =$$

$$= \left(\frac{1}{2}\right)^n \sigma[n] \cdot \left(1 + \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right)\right)$$

(ii) $$x[n] = \sum_{|z_k| < R_x} \operatorname{Res}_k \left\{ X(z) z^{n-1} \right\} = \operatorname{Res}_1 \left\{ X(z) z^{n-1} \right\} + \operatorname{Res}_2 \left\{ X(z) z^{n-1} \right\} + \operatorname{Res}_3 \left\{ X(z) z^{n-1} \right\}$$

$$\operatorname{Res}_1 \left\{ X(z) z^{n-1} \right\} = \frac{\frac{1}{8} \cdot \left(\frac{1}{2}\right)^{n-1}}{\frac{1}{4}} = \left(\frac{1}{2}\right)^n$$

$$\operatorname{Res}_2 \left\{ X(z) z^{n-1} \right\} = \frac{\frac{1}{8} \cdot e^{j\pi} \cdot \left(\frac{1}{2} e^{j\frac{\pi}{3}}\right)^{n-1}}{\frac{1}{2}\left(e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}\right) \cdot \frac{1}{2} e^{j\frac{2\pi}{3}}} = \frac{-\frac{1}{8} \cdot 2 \cdot \left(\frac{1}{2}\right)^n e^{j\frac{\pi}{3} n} \, e^{-j\frac{\pi}{3}}}{\frac{1}{2} \cdot \frac{\sqrt{3}}{2} e^{j\frac{\pi}{2}} \cdot e^{j\frac{2\pi}{3}}} = \frac{1}{2j} \frac{2}{\sqrt{3}} \cdot \left(\frac{1}{2}\right)^n e^{j\frac{\pi}{3} n}$$

$$\operatorname{Res}_3 \left\{ X(z) z^{n-1} \right\} = \frac{\frac{1}{8} e^{-j\pi} \cdot \left(\frac{1}{2} e^{-j\frac{\pi}{3}}\right)^{n-1}}{\frac{1}{2}\left(e^{-j\frac{\pi}{3}} - e^{j\frac{\pi}{3}}\right) \cdot \frac{1}{2} e^{-j\frac{2\pi}{3}}} = \frac{-\frac{1}{8} \cdot 2 \cdot \left(\frac{1}{2}\right)^n e^{-j\frac{\pi}{3} n} \, e^{j\frac{\pi}{3}}}{-\frac{1}{2} \frac{\sqrt{3}}{2} e^{j\frac{\pi}{2}} \cdot e^{-j\frac{2\pi}{3}}} = -\frac{1}{2j} \frac{2}{\sqrt{3}} \left(\frac{1}{2}\right)^n e^{-j\frac{\pi}{3} n}$$

$x[n]$ ... rechtsseitig: $\sigma[n]$!

$$x[n] = \left(\frac{1}{2}\right)^n \cdot \sigma[n] \cdot \left\{ 1 + \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right) \right\}$$