A 1.6

$x[n] \circ\!\!-\!\!\bullet\; c_k$

$y[n] \circ\!\!-\!\!\bullet\; d_k$

(a) $y[n] = x[n - \frac{N}{2}] \circ\!\!-\!\!\bullet\; e^{-j\frac{2\pi k}{N}\frac{N}{2}} c_k = e^{-j\pi k} c_k = (-1)^k c_k$

(b) $y[n] = x[-n+N] \circ\!\!-\!\!\bullet\; e^{-j\frac{2\pi k}{N}N} c_{-k} = e^{-j2\pi k} c_{-k} = c_{-k}$

(c) $y[n] = \frac{1}{2}\left(x[n] + x^*[N-n]\right) \circ\!\!-\!\!\bullet\; \frac{1}{2}\left(c_k + c_k^*\right) = \mathrm{Re}\{c_k\}$

(e) $y[n] = x[n] \cdot \cos\left(\frac{2\pi L}{M}\cdot n\right) = x[n] \cdot \frac{1}{2} \cdot \left\{ e^{j\frac{2\pi L}{M}n} + e^{-j\frac{2\pi L}{M}n} \right\} =$

$= \frac{1}{2} \cdot x[n] \cdot \left\{ e^{j\frac{2\pi}{N}\cdot\frac{L}{M}\cdot N\cdot n} + e^{-j\frac{2\pi}{N}\frac{L}{M}\cdot N\cdot n} \right\} = \frac{1}{2} \cdot \left\{ x[n]\, e^{j\frac{2\pi}{N}\frac{L}{M}\cdot N\cdot n} + x[n]\, e^{-j\frac{2\pi}{N}\frac{L}{M}\cdot N\cdot n} \right\} \circ\!\!-\!\!\bullet$

$\circ\!\!-\!\!\bullet\; \frac{1}{2} \cdot c_{k-\frac{L}{M}N} + \frac{1}{2}\, c_{k+\frac{L}{M}\cdot N}$

(f) $y[n] = x^2[n] = x[n] \cdot x[n] \circ\!\!-\!\!\bullet\; \sum_{\ell=0}^{N-1} c_\ell \cdot c_{k-\ell}$

$$d_k = \frac{1}{N_y}\sum_{n=0}^{N_y-1} x[2n]\, e^{-j\frac{2\pi}{N_y}k\cdot n} = \frac{1}{N_y}\sum_{n=0}^{N_y-1}\left(\sum_{\ell=0}^{N_x-1} c_\ell\, e^{j\frac{2\pi}{N_x}\cdot \ell\cdot 2n}\right) e^{-j\frac{2\pi}{N_y}\cdot k\cdot n} =$$

$$= \frac{1}{N_y}\sum_{\ell=0}^{N_x-1} c_\ell \sum_{n=0}^{N_y-1} e^{j\frac{2\pi}{N_x}\cdot 2n\cdot \ell}\; e^{-j\frac{2\pi}{N_y}\cdot k\cdot n} =$$

$$= \frac{1}{N_y}\sum_{\ell=0}^{N_x-1} c_\ell \sum_{n=0}^{N_y-1} e^{j2\pi\left(\frac{2\ell}{N_x}-\frac{k}{N_y}\right)\cdot n} = \frac{1}{N_y}\sum_{\ell=0}^{N_x-1} c_\ell \cdot \frac{1-e^{j2\pi\left(2\ell\frac{N_y}{N_x}-k\right)}}{1-e^{j2\pi\left(\frac{2\ell}{N_x}-\frac{k}{N_y}\right)}}$$

$N_x \in \mathbb{N}_u \Rightarrow N_y = N_x$

$$d_k = \frac{1}{N_x}\cdot\sum_{\ell=0}^{N_x-1} c_\ell \cdot \frac{1-e^{j2\pi(2\ell-k)}}{1-e^{j\frac{2\pi}{N_x}(2\ell-k)}}$$

$$\frac{2\pi}{N_x}(2\ell-k) = 2\pi\cdot m \quad , m\in\mathbb{Z} \quad (0,1)$$

$$\ell = \frac{k+m\cdot N_x}{2} = \frac{k}{2}+m\frac{N_x}{2}$$

$$\lim_{\ell\to\frac{k}{2}+m\frac{N_x}{2}} \frac{1-e^{j2\pi(2\ell-k)}}{1-e^{j\frac{2\pi}{N_x}(2\ell-k)}} = \left.\frac{-j2\pi\cdot 2\cdot e^{j2\pi(2\ell-k)}}{-j\frac{2\pi}{N_x}2\cdot e^{j\frac{2\pi}{N_x}(2\ell-k)}}\right|_{\ell=\frac{k}{2}+m\cdot\frac{N_x}{2}} = N_x$$

$$d_k = \begin{cases} c_{\frac{k}{2}} & k\in\mathbb{N}_g^0 \\ c_{\frac{k}{2}+\frac{N_x}{2}} & k\in\mathbb{N}_u \end{cases}$$

$N_x \in \mathbb{N}_g \Rightarrow N_y = \frac{N_x}{2}$

$$d_k = \frac{2}{N_x}\sum_{\ell=0}^{N_x-1} c_\ell\, \frac{1-e^{j2\pi(\ell-k)}}{1-e^{j\frac{2\pi}{N_x}2(\ell-k)}}$$

$$\frac{2\pi}{N_x}2(\ell-k) = 2\pi\cdot m \quad , \quad m\in\mathbb{Z}$$

$$\ell = k+m\cdot\frac{N_x}{2}$$

$$\lim_{\ell\to k+m\frac{N_x}{2}} \frac{1-e^{j2\pi(\ell-k)}}{1-e^{j\frac{2\pi}{N_x}2(\ell-k)}} = \left.\frac{-j2\pi\, e^{j2\pi(\ell-k)}}{-j\frac{2\pi}{N_x}\cdot 2\, e^{j\frac{2\pi}{N_x}2(\ell-k)}}\right|_{\ell=k+m\frac{N_x}{2}} = \frac{N_x}{2}$$

$$d_k = c_{k+\frac{N_x}{2}} + c_k$$