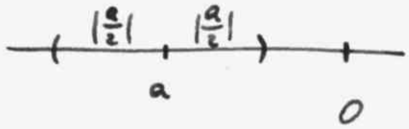
236. $\langle a_n \rangle = \langle 0,\ 0,1,\ 0,1,2,\ 0,1,2,3,\ \dots \rangle$

$a \in \mathbb{R},\quad a \notin \mathbb{N}$

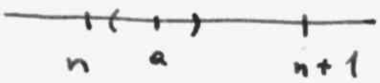
<u>Fall 1:</u>

$a < 0$



Es gibt eine Kugelumgebung um a, die kein Element von a_n enthält $\Rightarrow$ a ist kein HP (Radius $|\frac{a}{2}|$).

<u>Fall 2:</u> $n < a < n+1$; $n \in \mathbb{N}$



wie bei Fall 1 (Radius $\min\left(\frac{a-n}{2}, \frac{n+1-a}{2}\right)$).

$\Rightarrow$ nur die natürlichen Zahlen HP

237. $\langle a_n \rangle = \langle 0,\ 0,1,-1,\ 0,1,-1,2,-2,\ \dots \rangle$

$a \in \mathbb{R},\quad a \notin \mathbb{Z}$

$z < a < z+1$; $z \in \mathbb{Z}$

$r = \min\left(\frac{a-z}{2}, \frac{z+1-a}{2}\right)$

$\Rightarrow$ wie bei 236.

$\Rightarrow$ nur ganze Zahlen HP

238. ~~[illegible]~~ Jede irrationale Zahl lässt sich beliebig genau durch eine rationale Zahl annähern, d.h. in jeder Kugelumgebung um eine ~~[illegible]~~ irrationale Zahl ~~liegt auch~~ ~~mindestens eine~~ ~~rationale Zahlen~~ liegt auch mindestens eine rationale Zahl, d.h. es gibt keine Folge, die genau die rationalen Zahlen als HP hat.

240. $a_n = (-1)^n + \cos\frac{n\pi}{2} \qquad (n \geq 0)$

$n=0$: $a_0 = 1 + 1 = 2$
$n=1$: $a_1 = -1 + 0 = -1$
$n=2$: $a_2 = 1 - 1 = 0$
$n=3$: $a_3 = -1 + 0 = -1$

$a_n = a_{n+4}$

$$(-1)^n + \cos\frac{n\pi}{2} = (-1)^{n+4} + \cos\frac{(n+4)\pi}{2}$$

RS: $(-1)^{n+4} + \cos\frac{(n+4)\pi}{2} = (-1)^n \cdot (-1)^4 + \cos\left(\frac{n\pi}{2} + 2\pi\right) =$

$= (-1)^n + \cos\frac{n\pi}{2}$

LS = RS ✓

=> Häufungspunkte: $-1, 0, 2$

241. $a_n = \sin\frac{n\pi}{2} + (-1)^{\frac{n(n+1)}{2}} \qquad (n \geq 0)$

$n=0$: $a_0 = 0 + 1 = 1$
$n=1$: $a_1 = 1 - 1 = 0$
$n=2$: $a_2 = 0 - 1 = -1$
$n=3$: $a_3 = -1 + 1 = 0$

$a_n = a_{n+4}$

$$\sin\frac{n\pi}{2} + (-1)^{\frac{n(n+1)}{2}} = \sin\frac{(n+4)\pi}{2} + (-1)^{\frac{(n+4)(n+5)}{2}} =$$

RS: $\sin\left(\frac{n\pi}{2} + 2\pi\right) + (-1)^{\frac{n^2+9n+20}{2}} =$

$= \sin\frac{n\pi}{2} + (-1)^{\frac{n^2+n}{2}} \cdot (-1)^{4n+10} =$

$= \sin\frac{n\pi}{2} + (-1)^{\frac{n(n+1)}{2}}$

LS = RS ✓

-> Häufungspunkte: $1, 0, -1$

242.

$$a_n = \frac{\sin n}{n} \quad (n \geq 1)$$

$$a_n = \underbrace{\sin n}_{\text{beschränkte Folge}} \cdot \underbrace{\frac{1}{n}}_{\text{Nullfolge}}$$

$\Rightarrow$ a_n ist Nullfolge, 0 ist einzigen HP

243.

$$a_n = \frac{\sin n + \cos n}{\sqrt{n}} \quad (n \geq 1)$$

$$a_n = \underbrace{(\sin n + \cos n)}_{\text{beschränkte Folge}} \cdot \underbrace{\frac{1}{\sqrt{n}}}_{\text{Nullfolge}}$$

$\Rightarrow$ $a_n \in \mathcal{N}$, 0 ist einzigen HP

244.

$$a_n = \frac{\sin n + \cos n}{\sqrt{n}}$$

$$0 \leq |a_n| = \left| \frac{\sin n + \cos n}{\sqrt{n}} \right| \leq \frac{2}{\underset{+}{\sqrt{n}}} < \varepsilon$$

$$\underset{+}{\sqrt{n}} > \frac{2}{\varepsilon}$$

$$n > \frac{4}{\varepsilon^2} \quad \Rightarrow \quad N = N(\varepsilon) = \left\lceil \frac{4}{\varepsilon^2} \right\rceil$$

245.

$$a_n = \frac{\sin n}{\sqrt[4]{n}}$$

$$0 \leq |a_n| = \left| \frac{\sin n}{\sqrt[4]{n}} \right| \leq \frac{1}{\underset{+}{\sqrt[4]{n}}} < \varepsilon$$

$$\underset{+}{\sqrt[4]{n}} > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon^4} \quad \Rightarrow \quad N = N(\varepsilon) = \left\lceil \frac{1}{\varepsilon^4} \right\rceil$$

248. $\lim \langle a_n \rangle = a$

$\lim \langle b_n \rangle = b$

$\langle c_n \rangle = \langle a_n + 2b_n \rangle$

$\lim \langle c_n \rangle = a + 2b$

$|c_n - c| = |a_n + 2b_n - a - 2b| \leq |a_n - a| + |2b_n - 2b| < \varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

f.a. $n > N\left(\frac{\varepsilon}{2}\right), M\left(\frac{\varepsilon}{4}\right)$

$\underline{n > \max\left(N\left(\frac{\varepsilon}{2}\right), M\left(\frac{\varepsilon}{4}\right)\right)}$

249. $\lim \langle a_n \rangle = a$; $\lim \langle b_n \rangle = b$

$\langle c_n \rangle = \langle 3a_n - b_n \rangle$

$\lim c_n = 3a - b = c$

$|c_n - c| = |3a_n - b_n - 3a + b| \leq |3a_n - 3a| - |b_n - b| < \varepsilon = \frac{3\varepsilon}{2} - \frac{\varepsilon}{2}$

f.a. $n > N\left(\frac{\varepsilon}{2}\right), M\left(\frac{\varepsilon}{2}\right)$

$\underline{n > \max\left(N\left(\frac{\varepsilon}{2}\right), M\left(\frac{\varepsilon}{2}\right)\right)}$

250. $a_0 = 3$

$a_{n+1} = \sqrt{2a_n - 1}$ f.a. $n \geq 0$

$a_1 = \sqrt{5} > a_0$

Annahme: $a_n > a_{n+1}$

$2a_n > 2a_{n+1}$

$2a_n - 1 > 2a_{n+1} - 1$

$\sqrt{2a_n - 1} > \sqrt{2a_{n+1} - 1}$

$\underline{a_{n+1} > a_{n+2}}$ => streng monoton fallend

$a = \sqrt{2a - 1}$

$a^2 = 2a - 1$

$a^2 - 2a + 1 = 0$

$a_{1,2} = 1 \pm \sqrt{1 - 1}$

$\underline{a = 1}$ => Grenzwert: 1

=> $\langle a_n \rangle$ ist beschränkt.

251. $a_0 = 4$

$a_{n+1} = \sqrt{6a_n - 9}$ f.a. $n \geq 0$

$a_1 = \sqrt{15} < a_0$

Annahme: $a_{n+1} < a_n$

$a_{n+1} < a_n$

$6a_{n+1} < 6a_n$

$6a_{n+1} - 9 < 6a_n - 9$

$\sqrt{6a_{n+1} - 9} < \sqrt{6a_n - 9}$

$a_{n+2} < a_{n+1}$ => streng monoton fallend

$a = \sqrt{6a - 9}$

$a^2 - 6a + 9 = 0$

$a_{1,2} = 3 \pm \sqrt{9 - 9}$

$\underline{a = 3}$ => $\langle a_n \rangle$ beschränkt

Grenzwert: 3

252. $a_0 = 2$

$a_{n+1} = \sqrt{a_n + 1}$ f.a. $n \geq 0$

$a_1 = \sqrt{3} < a_0$

Annahme: $a_{n+1} < a_n$

$\sqrt{a_{n+1} + 1} < \sqrt{a_n + 1}$

$a_{n+2} < a_{n+1}$ => streng monoton fallend!

$a = \sqrt{a + 1}$

$a^2 - a - 1 = 0$

$a_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1 \pm \sqrt{5}}{2}$

=> $\langle a_n \rangle$ beschränkt.

Grenzwert: $\frac{1+\sqrt{5}}{2}$

253. $a_n = \dfrac{2n^3+2n-3}{4n^3+n^2+5} = \dfrac{2+\frac{2}{n^2}-\frac{3}{n^3}}{4+\frac{1}{n}+\frac{5}{n^3}}$

$\lim a_n = \frac{1}{2}$

254. $a_n = \dfrac{4n^2+5n-3}{2n^3+3n^2-n+7} = \dfrac{\frac{4}{n}+\frac{5}{n^2}-\frac{3}{n^3}}{2+\frac{3}{n}-\frac{1}{n^2}+\frac{7}{n^3}}$

$\lim a_n = 0$

255. $a_n = \dfrac{3n^2-5n+7}{3n^3-5n+7} = \dfrac{\frac{3}{n}-\frac{5}{n^2}+\frac{7}{n^3}}{3-\frac{5}{n^2}+\frac{7}{n^3}}$

$\lim a_n = 0$

256. $a_n = \dfrac{2n^3-5n^2+7}{2n^3-5n+7} = \dfrac{2-\frac{5}{n}+\frac{7}{n^3}}{2-\frac{5}{n^2}+\frac{7}{n^3}}$

$\lim a_n = 1$

257. $a_n = \dfrac{2n^2-5n^{\frac{9}{4}}+7}{7n^3+2n^{-\frac{3}{2}}+1} = \dfrac{\frac{2}{n}-\frac{15}{n^{\frac{3}{4}}}+\frac{7}{n^3}}{7+\frac{2}{n^{\frac{9}{2}}}+\frac{1}{n^3}}$

$\lim a_n = 0$

258. $a_n = \dfrac{3n^2-4n^{\frac{11}{3}}+n^{-1}}{2n^4+2n^{-\frac{3}{2}}+1} =$

$= \dfrac{\frac{3}{n^2}-\frac{4}{\sqrt[3]{n}}+\frac{1}{n^5}}{2+\frac{2}{n^{\frac{11}{2}}}+\frac{1}{n^4}}$

$\lim a_n = 0$

259. $a_n = \sqrt{n+1}-\sqrt{n} = \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

$\lim a_n = 0$

260.

$$a_n = \sqrt{n+\sqrt{n}} - \sqrt{n} = \frac{n+\sqrt{n}-n}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} =$$

$$= \frac{1}{\frac{\sqrt{n+\sqrt{n}}}{\sqrt{n}}+1}$$

$$b_n = \frac{\sqrt{n+\sqrt{n}}}{\sqrt{n}} = \frac{\sqrt{\sqrt{n}\cdot(\sqrt{n}+1)}}{\sqrt{n}} = \frac{\sqrt[4]{n}\cdot\sqrt{\sqrt{n}+1}}{\sqrt{n}} =$$

$$= \sqrt{n}\cdot\sqrt{\sqrt{n}+1}$$

$$\lim b_n = \infty$$

$$\Rightarrow \lim a_n = \frac{1}{\infty+1} = 0$$

261. $n! = 1 \cdot 2 \cdot \ldots \cdot (n-1)\cdot n$

$$n^n = n \cdot n \cdot \ldots \cdot n \cdot n$$

$$\Rightarrow \lim \frac{n!}{n^n} = 0$$

262.

$$a_n = \frac{\sqrt{n+2}-\sqrt{n}}{\sqrt[3]{\frac{1}{n}}} = \frac{n+2-n}{n^{-\frac{1}{3}}\cdot \cancel{\ldots}(\sqrt{n+2}+\sqrt{n})} = \frac{2}{n^{-\frac{1}{3}}\cdot(\sqrt{n+2}+\sqrt{n})}$$

$$\lim a_n = 0$$

263.

$$a_n = \frac{\frac{\sin n}{(n-2)^2}+\frac{n^2+2}{n^2-n}}{\frac{3n^2+2}{n^2+n}}$$

$$\lim a_n = \frac{1}{3}$$

264.

$$a_n = \frac{\frac{n^2-4}{4n^2-7n}-\frac{\cos n}{2n-5}}{\frac{3n^2+2}{(n-3)^2}}$$

$$\lim a_n = \frac{1}{12}$$

261.

$$\frac{n!}{n^n} = \underbrace{\frac{1}{n}}_{\text{Nullfolge}} \cdot \underbrace{\frac{2}{n} \cdot \frac{3}{n} \cdot \ldots \cdot \frac{n-1}{n} \cdot \frac{n}{n}}_{\text{jeweils } \leq 1 \text{, also beschränkt}}$$

daher: $\lim \frac{n!}{n^n} = 0$

264.
$$\frac{\frac{n^2-4}{4n^2-7n}-\frac{\cos n}{2n-5}}{\frac{3n^2+2}{(n-3)^2}} = a_n = \frac{\frac{1-\frac{4}{n^2}}{4-\frac{7}{n}} - \frac{\cos n}{2n-5}}{\frac{3+\frac{2}{n^2}}{1-\frac{6}{n}+\frac{9}{n^2}}}$$

$$\lim a_n = \frac{\frac{1-0}{4-0}-0}{\frac{3+0}{1-0+0}} = \frac{\frac{1}{4}}{3} = \frac{1}{12}$$

$n \in \mathbb{N} \setminus \{0,3\}$

265. $a_n = n \cdot q^n \qquad (-1<q<0)$

da $0<|q|<1$ gilt es ein $k>0 \in \mathbb{R}$, sodass $|q| = \frac{1}{1+k}$

Behauptung: $\lim a_n = 0$

$$|a_n - \lim a_n| = |a_n| = |n \cdot q^n| = n \cdot |q|^n = n \cdot \left(\frac{1}{1+k}\right)^n =$$

$$= \frac{n}{(1+k)^n} \leq \frac{n}{1+n\cdot k+\frac{n(n-1)}{2}k^2} \leq \frac{n}{\frac{n(n-1)}{2}k^2} = \frac{2}{(n-1)k^2} < \varepsilon$$

$$\frac{2}{\varepsilon} < (n-1)k^2 \qquad \frac{2}{\varepsilon k^2}+1 < n$$

D. h. f. a. $n > N = N(\varepsilon) = \frac{2}{\varepsilon k^2}+1$ gilt: $|a_n| < \varepsilon$, also $\lim a_n = 0$.

266. $a_n = \frac{q^n}{n} \qquad (q>1)$

da $q>1$, gilt es ein k mit $0<k<1$, sodass $|q| = \frac{1}{k}$

$$\lim a_n = \lim \frac{q^n}{n} = \lim \underbrace{\frac{1}{nk^n}}_{\text{Nullfolge (siehe 265.)}} = +\infty$$

267. $a_n = \sqrt[n^2]{n^5+1}$

$b_n = \sqrt[n^2]{n^2} < a_n$ f.a. $n>0$; $\lim b_n = 1$

$c_n = \sqrt[n^2]{n^6} > a_n$ f.a. $n>1$

$\lim \sqrt[n^2]{n^6} = \lim \left(\sqrt[n^2]{n^2}\right)^3 = \left(\lim \sqrt[n^2]{n^2}\right)^3 = 1^3 = 1$

daher: $\lim b_n \leq \lim a_n \leq \lim c_n$

$\underline{\lim a_n = 1}$

268. $a_n = \sqrt[n^2]{n^3+n^2}$

$b_n = \sqrt[n^2]{n^2} < a_n$ f.a. $n>0$; $\lim b_n = 1$

$c_n = \sqrt[n^2]{n^4} > a_n$ f.f.a. n

$\lim \sqrt[n^2]{n^4} = \lim \left(\sqrt[n^2]{n^2}\right)^4 = \left(\lim \sqrt[n^2]{n^2}\right)^2 = 1$

$\Rightarrow \underline{\lim a_n = 1}$

269. $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \ldots + \frac{1}{n^2+n}$

$b_n = \frac{n}{n^2+n} < a_n$ f.f.a. n; $\lim b_n = 0$

$c_n = \frac{n}{n^2+1} > a_n$ f.f.a. n; $\lim c_n = 0$

daher: $\underline{\lim a_n = 0}$

270. $a_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+n)^2}$

$b_n = \frac{n}{4n^2} < a_n$ f.f.a. n; $\lim b_n = 0$

$c_n = \frac{n}{(n+1)^2} > a_n$ f.f.a. n; $\lim c_n = 0$

daher: $\underline{\lim a_n = 0}$

271. $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$

$b_n = \frac{n}{\sqrt{n^2+n}} = \sqrt{\frac{n^2}{n^2+n}} < a_n$ f.f.a. n; $\lim b_n = 1$

$c_n = \frac{n}{\sqrt{n^2+1}} = \sqrt{\frac{n^2}{n^2+1}} > a_n$ f.f.a. n; $\lim c_n = 1$

daher $\underline{\lim a_n = 1}$

272. $a_n = \frac{n^2+1}{n^3+1} + \frac{n^2+2}{n^3+2} + \ldots + \frac{n^2+n}{n^3+n}$

$b_n = n \cdot \frac{n^2+1}{n^3+n} = \frac{n^3+n}{n^3+n} = 1 < a_n$ f.f.a. n; $\lim b_n = 1$

$c_n = n \cdot \frac{n^2+n}{n^3+1} = \frac{n^3+n}{n^3+1} > a_n$ f.f.a. n; $\lim c_n = 1$

daher $\underline{\lim a_n = 1}$

274. $a_n = a_{n-1} + \frac{1}{n(n+1)} \quad (n \geq 1)$; $a_0 = 0$

$b_n = 1 - \frac{1}{n+1}$

Behauptung: $a_n = b_n$ f.a. n (Induktionsvoraussetzung)

Induktionsstart: $n = 0$; $a_0 = b_0$?

$a_0 = 0$; $b_0 = 1 - \frac{1}{1}$, $a_0 = b_0$; ~~Ihm~~ Behauptung wahr f. $n=0$

Induktionsschritt: $a_{n+1} = b_{n+1}$?

$a_{n+1} = a_n + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} =$

$= 1 - \frac{n+2-1}{(n+1)(n+2)} = 1 - \frac{n+1}{(n+1)(n+2)} = 1 - \frac{1}{n+2}$

$b_{n+1} = 1 - \frac{1}{n+2}$

$a_{n+1} = b_{n+1}$, also Behauptung wahr f.a. $n \in \mathbb{N}$

$\lim a_n = \lim b_n = \lim\left(1 - \frac{1}{n+1}\right) = 1$

275.

$$a_{n+1} = a_n + \frac{n}{(n+1)!} \qquad (n \geq 0) \qquad a_0 = 0$$

$$b_n = 1 - \frac{1}{n!}$$

Induktionsbehauptung: $a_n = b_n$

Induktionsstart: $n = 0$

$a_0 = 0$; $b_0 = 0$; $a_0 = b_0$, also Behauptung wahr f. $n=0$

Induktionsschritt: gilt $a_{n+1} = b_{n+1}$?

$$a_{n+1} = a_n + \frac{n}{(n+1)!} = 1 - \frac{1}{n!} + \frac{n}{(n+1)!} = 1 - \frac{n+1-n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

$$b_{n+1} = 1 - \frac{1}{(n+1)!}$$

$a_{n+1} = b_{n+1}$, also Behauptung wahr f.a. $n \in \mathbb{N}$

$$\lim a_n = \lim b_n = \lim \left(1 - \frac{1}{n!}\right) = 1$$

279.

$$a_n = \frac{2n^4+n}{n^3+n} = \frac{2n^3+1}{n^2+1} \qquad (n \neq 0)$$

$$a_n > \frac{2n^4+n}{n^3+n^3} = \frac{2n^4+n}{2n^3} > \frac{2n^4}{2n^3} = n > A$$

$\underline{N(A) = A}$

also unbestimmt konvergent

280. $a_n = \frac{1}{n^p}$; $b_n = \frac{1}{n^q}$; $q < p < 2q$

$$\lim \frac{a_n}{b_n} = \lim \frac{n^q}{n^p} = 0 \quad ; \quad \lim \frac{a_n}{b_n^2} = \lim \frac{n^{2q}}{n^p} = +\infty$$

$$\lim a_n = \lim b_n = 0$$

276. $a_n = (-1)^n n^{\left((-1)^{\frac{n(n+1)}{2}}+1\right)} + \cos\frac{n\pi}{2}$

Zerlegung in 4 TF:

$n=4k$: $a_n = n^2+1$; $\lim a_n = +\infty$

$n=4k+1$: $a_n = -n^0 + 0 = -1$; $\lim a_n = -1$

$n=4k+2$: $a_n = n^0 - 1 = 0$; $\lim a_n = 0$

$n=4k+3$: $a_n = -n^2 + 0 = -n^2$; $\lim a_n = -\infty$

HP: $-\infty, -1, 0, +\infty$; $\underline{\lim}\, a_n = -\infty$, $\overline{\lim}\, a_n = +\infty$

277. $a_n = \dfrac{n^2\cos\frac{n\pi}{2}+1}{n+1} + \sin\dfrac{(2n+1)\pi}{2}$

Zerlegung in 4 TF:

$n=4k$: $a_n = \dfrac{n^2+1}{n+1} + 1$; $\lim a_n = +\infty$

$n=4k+1$: $a_n = \dfrac{1}{n+1} - 1$; $\lim a_n = -1$

$n=4k+2$: $a_n = \dfrac{-n^2+1}{n+1} + 1$; $\lim a_n = -\infty$

$n=4k+3$: $a_n = \dfrac{1}{n+1} - 1$; $\lim a_n = -1$

HP: $-\infty, -1, +\infty$; $\underline{\lim}\, a_n = -\infty$; $\overline{\lim}\, a_n = +\infty$

278. $a_n = \dfrac{n^3+1}{n-1}$; $n \neq 1$

$a_n > A$, also $\dfrac{n^3+1}{n-1} > A$

$\dfrac{n^3+1}{n-1} > \dfrac{n^3+1}{n} > \dfrac{n^3}{n} = n^2 > A$ $(n \neq 0)$

$n > \sqrt{A}$

$\underline{N(A) = \sqrt{A}}$

also divergent (bestimmt)

281. $a_n = n^p$; $b_n = n^q$; $p < q < 2p$

$$\lim \frac{a_n}{b_n} = \lim \frac{n^p}{n^q} = 0 \quad ; \quad \lim \frac{a_n^2}{b_n} = \lim \frac{n^{2p}}{n^q} = +\infty$$

$$\lim a_n = \lim b_n = +\infty$$

282. $$\frac{3}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2)+Bn}{n(n+2)} = \frac{An+Bn+2A}{n(n+2)}$$

$$A+B=0$$

$$2A = 3 \qquad A = \frac{3}{2}, \ B = -\frac{3}{2}$$

$$s_k = \sum_{n=1}^{k} \frac{3}{n(n+2)} = \sum_{n=1}^{k} \left(\frac{3}{2n} - \frac{3}{2(n+2)} \right) =$$

$$= \sum_{n=1}^{k} \frac{3}{2n} - \sum_{n=1}^{k} \frac{3}{2(n+2)} = \sum_{n=1}^{k} \frac{3}{2n} - \sum_{n=3}^{k+2} \frac{3}{2n} =$$

$$= \frac{3}{2} + \frac{3}{4} - \frac{3}{2(k+1)} - \frac{3}{2(k+2)} + \sum_{n=3}^{k} \frac{3}{2n} - \sum_{n=3}^{k} \frac{3}{2n} =$$

$$= \frac{3}{2} + \frac{3}{4} - \frac{3}{2(k+1)} - \frac{3}{2(k+2)}$$

$$\sum_{n=1}^{k} \frac{3}{n(n+2)} = \lim s_k = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$

283. $$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)} = \frac{(A+B)n+A}{n(n+1)}$$

$$A+B=0; \quad A=1; \quad B=-1$$

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{k} \frac{1}{n} - \sum_{n=1}^{k} \frac{1}{n+1} =$$

$$= \sum_{n=1}^{k} \frac{1}{n} - \sum_{n=2}^{k+1} \frac{1}{n} = \frac{1}{1} + \sum_{n=2}^{k} \frac{1}{n} - \sum_{n=2}^{k} \frac{1}{n} - \frac{1}{k+1} = 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \lim s_k = 1$$

284. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

$$\frac{n}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$s_k = \sum_{n=1}^{k} \frac{n}{(n+1)!} = \sum_{n=1}^{k} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) = \sum_{n=1}^{k} \frac{1}{n!} - \sum_{n=1}^{k} \frac{1}{(n+1)!} =$$

$$= \sum_{n=1}^{k} \frac{1}{n!} - \sum_{n=2}^{k+1} \frac{1}{n!} = \frac{1}{1} + \sum_{n=2}^{k} \frac{1}{n!} - \sum_{n=2}^{k} \frac{1}{n!} - \frac{1}{(k+1)!} =$$

$$= 1 - \frac{1}{(k+1)!}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim s_k = 1$$

285. $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)!}$

$$\frac{n+1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$s_k = \sum_{n=1}^{k} \frac{n+1}{(n+2)!} = \sum_{n=1}^{k} \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right) =$$

$$= \sum_{n=2}^{k+1} \frac{1}{n!} - \sum_{n=3}^{k+2} \frac{1}{n!} = \frac{1}{2} + \sum_{n=3}^{k+1} \frac{1}{n!} - \sum_{n=3}^{k+1} \frac{1}{n!} - \frac{1}{(k+2)!} =$$

$$= \frac{1}{2} - \frac{1}{(k+2)!}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+2)!} = \lim s_k = \frac{1}{2}$$

286. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n+1}{n(n+1)}$

$$\frac{2n+1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)} = \frac{(A+B)n+A}{n(n+1)}$$

$A = B = 1$

$$s_k = \sum_{n=1}^{k} (-1)^n \left(\frac{1}{n} + \frac{1}{n+1}\right) =$$

$$= \sum_{n=1}^{k} (-1)^n \frac{1}{n} + \sum_{n=1}^{k} (-1)^n \frac{1}{n+1} = -1 + \sum_{n=2}^{k} (-1)^n \frac{1}{n} + \sum_{n=2}^{k} (-1)^{n-1} \frac{1}{n} + (-1)^{k} \frac{1}{k+1} =$$

$$= -1 + (-1)^{k} \cdot \frac{1}{k+1}$$

$$\lim s_k = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n+1}{n(n+1)} = -1$$

287. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n+5}{(n+2)(n+3)}$

$$\frac{2n+5}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3} = \frac{A(n+3)+B(n+2)}{(n+2)(n+3)} = \frac{n(A+B)+(3A+2B)}{(n+2)(n+3)}$$

$A+B=2;\ 3A+2B=5;\ A=B=1$

$$s_k = \sum_{n=1}^{k} (-1)^n \cdot \frac{2n+5}{(n+2)(n+3)} = \sum_{n=1}^{k} (-1)^n \cdot \left(\frac{1}{n+2} + \frac{1}{n+3}\right) =$$

$$= \sum_{n=1}^{k} (-1)^n \frac{1}{n+2} + \sum_{n=1}^{k} (-1)^n \frac{1}{n+3} = \sum_{n=1}^{k} (-1)^n \frac{1}{n+2} + \sum_{n=2}^{k+1} (-1)^{n-1} \frac{1}{n+2} =$$

$$= -\frac{1}{3} + \sum_{n=2}^{k} (-1)^n \frac{1}{n+2} + \sum_{n=2}^{k} (-1)^{n-1} \frac{1}{n+2} + (-1)^{k} \frac{1}{k+3} =$$

$$= -\frac{1}{3} + (-1)^{k} \frac{1}{k+3}$$

$$\lim s_k = \sum_{n=1}^{\infty} (-1)^n \frac{2n+5}{(n+2)(n+3)} = -\frac{1}{3}$$

288.

$$\sum_{n\geq 0} \frac{3n^2+1}{5n^3-2}$$

$$\frac{3n^2+1}{5n^3-2} > \frac{3n^2+1}{6n^3} > \frac{3n^2}{6n^3} = \frac{1}{2n} \quad \text{f.f.a.n}$$

Daher: $0 < \frac{1}{2n} < \frac{3n^2+1}{5n^3 \cancel{+} -2}$ f.f.a.n

also $\sum_{n\geq 1} \frac{1}{2n}$ ist Minorante von $\sum_{n\geq 0} \frac{3n^2+1}{5n^3-2}$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \lim\left(\sum_{n=1}^{k} \frac{1}{2n}\right) = \lim\left(\frac{1}{2}\cdot\sum_{n=1}^{k} \frac{1}{n}\right) =$$

$$= \lim \frac{1}{2} \cdot \lim\left(\sum_{n=1}^{k} \frac{1}{n}\right) = \frac{1}{2}\cdot(+\infty) = +\infty$$

$\sum_{n\geq 1} \frac{1}{2n}$ ist divergent $\Rightarrow \sum_{n\geq 0} \frac{3n^2+1}{5n^3-2}$ ist divergent

289. $\sum_{n\geq 0} \frac{n-2}{2n^3+5n-3}$

$$\cancel{\frac{n-2}{2n^3+5n-3} > \frac{\frac{n}{2}}{2n^3+5n-3} = \frac{n}{4n^3+10n-6} > \frac{n}{5n^3} = \frac{1}{5n^2} \quad \text{f.f.a.n}}$$

~~f.f.a. gilt: ... a~~

$$\frac{n-2}{2n^3+5n-3} < \frac{n}{2n^3+5n-3} < \frac{n}{2n^3} = \frac{1}{2n^2} \quad \text{f.f.a.n}$$

$$\frac{n-2}{2n^3+5n-3} < \frac{1}{2n^2} \quad \text{f.f.a.n}$$

Daher: $\sum_{n\geq 1} \frac{1}{2n^2}$ ist Majorante von $\sum \frac{n-2}{2n^3+5n-3}$

$$\sum_{n=1}^{\infty} \frac{1}{2n^2} = \lim\left(\sum_{n=1}^{k} \frac{1}{2n^2}\right) = \frac{1}{2}\cdot \underbrace{\lim\left(\sum_{n=1}^{k} \frac{1}{n^2}\right)}_{\neq +\infty \text{ (hyperharmonische Reihe)}} \quad \text{existiert}$$

$\sum_{n\geq 1} \frac{1}{2n^2}$ ist konvergent $\Rightarrow \sum_{n\geq 0} \frac{n-2}{2n^3+5n-3}$ ist konvergent

290. $\sum_{n\geq 0} \frac{n+2}{6^n}$ $\qquad a_n = \frac{n+2}{6^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{n+3}{6^{n+1}}}{\frac{n+2}{6^n}}\right| = \left|\frac{n+3}{n+2}\cdot\frac{6^n}{6^{n+1}}\right| =$$

$$= \left|\frac{n+3}{n+2}\right|\cdot\frac{1}{6} = \frac{n+3}{n+2}\cdot\frac{1}{6} < \qquad (n\geq 0)$$

$$< \frac{2n+4}{n+2}\cdot\frac{1}{6} = 2\cdot\frac{1}{6} = \frac{1}{3} < 1$$

$$\overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{3}$$

D.h. $\sum_{n\geq 0}\frac{n+2}{6^n}$ konvergiert.

291. $\sum_{n\geq 1}\frac{n!}{n^n}$ $\qquad a_n = \frac{n!}{n^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \left|\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}\right| =$$

$$= \left|(n+1)\frac{n^n}{(n+1)^n}\cdot\frac{1}{n+1}\right| = \left|\frac{n^n}{(n+1)^n}\right| =$$

$$= \left|\left(\frac{n}{n+1}\right)^n\right| = \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$$

$$\underbrace{\left(1+\frac{1}{n}\right)^n} \geq 2 > 0$$

$$\left(\frac{n+1}{n}\right)^n \geq 2$$

$$\frac{1}{2} \geq \left(\frac{n}{n+1}\right)^n$$

$$\overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1}{2}$$

D.h. $\sum_{n\geq 1}\frac{n!}{n^n}$ konvergiert.

292. $\sum\limits_{n\geq 0} \frac{2n^2+1}{n^4+2}$

$$\cancel{\frac{2n^2+1}{n^4+2} < \frac{2n^2+1}{4n^4+2} = \frac{1}{2}}$$

$$\cancel{\frac{2n^2+1}{n^4+2} < \frac{2n^2+1}{2n^4+n^2} = \frac{1}{n^2}} \quad \text{~~f.f.a.n~~}$$

~~f.f.a.n gilt:~~ $\cancel{0 < \frac{2n^2+1}{n^4+2} \leq \frac{1}{n^2}}$

~~D.h.~~ $\cancel{\sum\limits_{n\geq 1} \frac{1}{n^2}}$ ~~ist Majorante von~~ $\cancel{\sum\limits_{n\geq 0} \frac{2n^2+1}{n^4+2}}$

$\cancel{\sum\limits_{n\geq 1} \frac{1}{n^2}}$ ~~konvergiert, daher konvergiert auch~~ $\cancel{\sum\limits_{n\geq 0} \frac{2n^2+1}{n^4+2}}$

$$\cancel{\frac{2n^2+1}{n^4+2} < \frac{2n^2+2}{n^4+2} = \frac{2(n^2+1)}{n^4+2}}$$

$$\frac{2n^2+1}{n^4+2} < \cancel{\frac{2n^2+1}{2n^4}} \; \frac{2n^2+1}{n^4} < \cancel{4} \; \cancel{\frac{3n^2}{n^4} = 3 \cdot \frac{1}{n}}$$

$$< \frac{3n^2}{n^4} = 3 \cdot \frac{1}{n^2} \quad \text{f.f.a.n}$$

D.h. $\sum\limits_{n\geq 1} 3 \cdot \frac{1}{n^2}$ ist Majorante von $\sum\limits_{n\geq 0} \frac{2n^2+1}{n^4+2}$

$$\lim \sum_{n\geq 1}^{k} 3 \cdot \frac{1}{n^2} = \lim 3 \cdot \sum_{n=1}^{k} \frac{1}{n^2} =$$

$$= \lim 3 \cdot \lim \sum_{n=1}^{k} \frac{1}{n^2} = 3 \cdot \underbrace{\sum_{n\geq 1} \frac{1}{n^2}}_{\text{konvergent}}$$

D.h. $\sum\limits_{n\geq 0} \frac{2n^2+1}{n^4+2}$ ist auch konvergent.

293. $\sum_{n\geq 0} \frac{n+3}{7n^2-2n-1}$

$$\frac{n+3}{7n^2-2n-1} > \frac{n+3}{7n^2} > \frac{n}{7n^2} = \frac{1}{7}\cdot\frac{1}{n} \text{ f.f.a.n}$$

f.f.a.n gilt $\frac{1}{7n} < \frac{n+3}{7n^2-2n-1}$

D.h. $\sum_{n\geq 1} \frac{1}{7n}$ ist Minorante von $\sum_{n\geq 0} \frac{n+3}{7n^2-2n-1}$

$$\lim \sum_{n=1}^{k} \frac{1}{7n} = \frac{1}{7}\cdot \lim \sum_{n=1}^{k} \frac{1}{n} =$$

$$= \frac{1}{7}\cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{+\infty} = +\infty$$

D.h. $\sum_{n\geq 0} \frac{n+3}{7n^2-2n-1} = +\infty$ (divergent)

294. $\sum_{n\geq 0} \frac{n-1}{3^n}$ $\qquad a_n = \frac{n-1}{3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n}{3^{n+1}}\cdot\frac{3^n}{n-1}\right| = \left|\frac{n}{3(n-1)}\right| = \frac{n}{3(n-1)} =$$

$$= \frac{n}{3n-3} < \frac{n}{n} = 1 \text{ f.f.a.n}$$

$$\overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

D.h. $\sum_{n\geq 0} \frac{n-1}{3^n}$ ist konvergent.

295. $\sum\limits_{n\geq 1} \frac{n^{n-1}}{n!}$ $\qquad a_n = \frac{n^{n-1}}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^n}{(n+1)!}\cdot\frac{n!}{n^{n-1}}\right| = \left|\frac{(n+1)^n}{n+1}\cdot\frac{1}{n^{n-1}}\right| =$$

$$= \left|\frac{(n+1)^{n-1}}{1}\cdot\frac{1}{n^{n-1}}\right| = \left|\left(\frac{n+1}{n}\right)^{n-1}\right| = \left(\frac{n+1}{n}\right)^{n-1} =$$

$$= \left(1+\frac{1}{n}\right)^{n-1} > 1 \quad \text{f.f.a. } n$$

D.h. $\sum\limits_{n\geq 1} \frac{n^{n-1}}{n!} = +\infty$ (divergent)

298. $\sum\limits_{n\geq 0} \frac{(-1)^n}{\sqrt{n^2+2}} = \sum\limits_{n\geq 0} (-1)^n \cdot \underbrace{\frac{1}{\sqrt{n^2+2}}}_{b_n}$

$\frac{1}{\sqrt{n^2+2}}$ ist monoton fallende Nullfolge

LEIBNIZ'sches Konvergenzkriterium:

$\sum\limits_{n\geq 0} \frac{(-1)^n}{\sqrt{n^2+2}}$ ist konvergent

299. $\sum\limits_{n\geq 0} \frac{(-1)^n}{n^{\frac{3}{2}}+5n} = \sum\limits_{n\geq 0} (-1)^n \cdot \underbrace{\frac{1}{\sqrt{n^3}+5n}}_{\searrow 0}$

$\Rightarrow \sum\limits_{n\geq 0} \frac{(-1)^n}{n^{\frac{3}{2}}+5n}$ ist konvergent

300. $\sum\limits_{n\geq 0} \frac{(-1)^n}{\sqrt[3]{n+2}} = \sum\limits_{n\geq 0} (-1)^n \cdot \underbrace{\frac{1}{\sqrt[3]{n+2}}}_{\searrow 0} \Rightarrow$ konvergent

301. $\sum\limits_{n\geq 0} \frac{(-1)^n}{(n+3)^{\frac{4}{3}}} = \sum\limits_{n\geq 0} (-1)^n \cdot \underbrace{\frac{1}{\sqrt[3]{(n+3)^4}}}_{\searrow 0} \rightarrow$ konvergent

302.

$$\overline{\lim} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\overline{\lim} \left| \frac{a_{n+1}^2}{a_n^2} \right| < 1 \quad ?$$

$$\underbrace{\overline{\lim} \left| \frac{a_{n+1}^2}{a_n^2} \right|}_{} $$

$$\overline{\lim} \left| \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+1}}{a_n} \right| \cancel{dh} = \underbrace{\underbrace{\overline{\lim} \left| \frac{a_{n+1}}{a_n} \right|}_{<1} \cdot \underbrace{\overline{\lim} \left| \frac{a_{n+1}}{a_n} \right|}_{<1}}_{<1}$$

D.h. aus $\sum_{n\geq 0} a_n$ konvergent ($a_n \geq 0$) folgt

$\sum_{n\geq 0} a_n^2$ konvergent

303. wie **302.**, nur ohne $a_n \geq 0$:

f.a. $a_n = 0$: $\sum_{n\geq 0} a_n$ sicher konvergent,

auch $\sum_{n\geq 0} a_n^2$ sicher konvergent

f. $a_n \neq 0$: analog zu **302.**

304. wie **302.**

305. wie **303.**

306. $\lim a_n = a$

$$\sum_{n\geq 0} (a_{n+1} - a_n)$$

$$s_k = \sum_{n=0}^{k} (a_{n+1} - a_n) = \sum_{n=0}^{k} a_{n+1} - \sum_{n=0}^{k} a_n = \sum_{n=1}^{k+1} a_n - \sum_{n=0}^{k} a_n =$$

$$= a_{k+1} + \sum_{n=1}^{k} a_n - \sum_{n=1}^{k} a_n - a_0 = a_{k+1} - a_0$$

$$\sum_{n\geq 0} (a_{n+1} - a_n) = \lim s_k = \lim (a_{k+1} - a_0) = \lim a_{k+1} - \lim a_0 =$$

$$= a - a_0$$

□

307. $\lim a_n = a$

$$\sum_{n\geq 0} (a_{n+2} - a_n) =$$

$$s_k = \sum_{n=0}^{k} (a_{n+2} - a_n) = \sum_{n=0}^{k} a_{n+2} - \sum_{n=0}^{k} a_n = \sum_{n=2}^{k+2} a_n - \sum_{n=0}^{k} a_n =$$

$$= \sum_{n=2}^{k} a_n + a_{k+1} + a_{k+2} - \sum_{n=2}^{k} a_n - a_0 - a_1 =$$

$$= a_{k+1} + a_{k+2} - a_0 - a_1$$

$$\sum_{n\geq 0} (a_{n+2} - a_n) = \lim s_k = \lim (a_{k+1} + a_{k+2} - a_0 - a_1) =$$

$$= 2a - a_0 - a_1$$

308. $\lim a_n = 0$

$$\sum_{n=0}^{\infty} (-1)^n (a_{n+1} + a_n)$$

$$s_k = \sum_{n=0}^{k} (-1)^n (a_{n+1} + a_n) = \sum_{n=0}^{k} (-1)^n a_{n+1} + \sum_{n=0}^{k} (-1)^n \cdot a_n =$$

$$= \sum_{n=1}^{k+1} (-1)^{n-1} a_n + \sum_{n=0}^{k} (-1)^n \cdot a_n =$$

$$= \sum_{n=1}^{k+1} (-1)^{n-1} a_n \cancel{+} - \sum_{n=0}^{k} (-1)^{n-1} \cdot a_n =$$

$$= (-1)^k \cdot a_{k+1} + \sum_{n=1}^{k} (-1)^{n-1} a_n \cancel{+} - (-1)^{-1} \cdot a_0 \cancel{+} - \sum_{n=1}^{k} (-1)^{n-1} a_n =$$

$$= (-1)^k \cdot a_{k+1} - (-1)^{-1} \cdot a_0 = a_0 + (-1)^k \cdot a_{k+1}$$

$$\sum_{n=0}^{\infty} (-1)^n (a_{n+1} + a_n) = \lim s_k = \lim (a_0 + \underbrace{(-1)^k}_{\text{beschränkt}} \cdot \underbrace{a_{k+1}}_{\downarrow 0}) =$$

$$= a_0$$

297.

$$\frac{\cos\frac{n\pi}{3}}{2^n} = \frac{\mathrm{Re}\left(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3}\right)^n}{2^n} =$$

$$= \mathrm{Re}\left(\frac{1}{4} + i\,\frac{\sqrt{3}}{4}\right)^n$$

$$\sum_{n\geq 0} \frac{\cos\frac{n\pi}{3}}{2^n}$$

$$s_k = \sum_{n=0}^{k} \frac{\cos\frac{n\pi}{3}}{2^n} = \mathrm{Re}\left(\sum_{n=0}^{k} \underbrace{\left(\frac{1}{4} + i\,\frac{\sqrt{3}}{4}\right)}_{q}{}^n\right)$$

$$|q| = \left|\frac{1}{4} + i\,\frac{\sqrt{3}}{4}\right| < 1$$

$$|q| = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\mathrm{Re}\left(\sum_{n=0}^{\infty}\left(\frac{1}{4} + i\,\frac{\sqrt{3}}{4}\right)^n\right) = \mathrm{Re}\left(\frac{1}{1 - \frac{1}{4} - i\frac{\sqrt{3}}{4}}\right) =$$

$$= \mathrm{Re}\left(\frac{4}{4 - 1 - i\cdot\sqrt{3}}\right) = \mathrm{Re}\left(\frac{4}{3 - i\cdot\sqrt{3}}\right) =$$

$$= \mathrm{Re}\left(\frac{4(3 + i\sqrt{3})}{9+3}\right) = \mathrm{Re}\left(\frac{12 + i\,4\cdot\sqrt{3}}{12}\right) =$$

$$= 1$$

296. $$\sum_{n\geq 0} \frac{\sin\frac{n\pi}{3}}{2^n} = \mathrm{Im}\left(\sum_{n\geq 0}\left(\frac{1}{4} + i\,\frac{\sqrt{3}}{4}\right)^n\right) =$$

$$= \mathrm{Im}\left(\frac{12 + i\,4\sqrt{3}}{12}\right) = \frac{\sqrt{3}}{3}$$

ansonsten wie 297.

309. $\sum\limits_{n\geq 0} \underbrace{\binom{\frac{1}{2}}{n} x^n}_{a_n} \qquad |x|<1$

Quotientenkriterium in Limesform:

$$\left|\frac{a_{k+1}}{a_k}\right| = \cancel{\frac{(k+1)(k)(k-1)\cdot\ldots}{}}$$

$$= \left| \frac{\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\ldots\cdot\left(\frac{1}{2}-n\right)}{(n+1)(n)(n-1)\cdot\ldots\cdot 1}\cdot x^{n+1}}{\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\ldots\cdot\left(\frac{1}{2}-n+1\right)}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1}\, x^n} \right| = \left| \frac{\frac{\frac{1}{2}-n}{n+1}\cdot x^{n+1}}{1\cdot x^n} \right| =$$

$$= \left| \frac{\frac{1}{2}-n}{n+1}\, x \right|$$

$$\overline{\lim} \left| \frac{\frac{1}{2}-n}{n+1}\, x \right| = \lim \left| \frac{\frac{1}{2}-n}{n+1}\, x \right| = |x| < 1$$

D.h. Reihe ist konvergent für $|x|<1$

310. $\sum\limits_{n\geq 0} \binom{2n}{n} x^n \qquad |x|<\frac{1}{4}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left| \frac{\frac{(2n+2)(2n+1)\cdot\ldots\cdot(n+2)}{(n+1)n(n-1)\cdot\ldots\cdot 1}}{\frac{2n(2n-1)\cdot\ldots\cdot(n+1)}{n(n-1)\cdot\ldots\cdot 1}}\, x \right| = \left| \frac{(2n+2)(2n+1)}{(n+1)^2}\, x \right| =$$

$$= \left| \frac{4n^2+6n+2}{(n+1)^2}\, x \right| = \left| \frac{4n^2+6n+2}{n^2+2n+1}\, x \right|$$

$$\overline{\lim} \left| \frac{4n^2+6n+2}{n^2+2n+1}\, x \right| = \lim |\ldots| = |4x| < 1 \iff |x| < \frac{1}{4}$$

D.h. für $|x|<\frac{1}{4}$ konvergiert die Reihe.

311. $\sum_{n\geq 0} \underbrace{\frac{z^{2n+1}}{(2n+1)!}}_{a_n}$, $z \in \mathbb{C}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{z^{2n+3}}{(2n+3)!}}{\frac{z^{2n+1}}{(2n+1)!}}\right| = \left|\frac{z^2}{(2n+3)(2n+2)}\right| \longrightarrow 0$$

D.h. die Reihe konvergiert f.a. $z \in \mathbb{C}$

312. $\sum_{n\geq 0} \frac{z^{2n}}{(2n)!}$, $z \in \mathbb{C}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{z^{2n+2}}{(2n+2)!}}{\frac{z^{2n}}{(2n)!}}\right| = \left|\frac{z^2}{(2n+1)(2n+2)}\right| \longrightarrow 0$$

D.h. die Reihe konvergiert f.a. $z \in \mathbb{C}$

313. $\sum_{n=1}^{\infty} \underbrace{\frac{(x-1)^n}{2n-1}}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{2n+1}}{\frac{(x-1)^{n}}{2n-1}}\right| = \left|\frac{(2n-1)(x-1)}{2n+1}\right| =$$

$$= \left|\frac{2nx - x + 1 - 2n}{2n+1}\right|$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{2n(x-1)-x+1}{2n+1}\right| = \left|\frac{2(x-1)}{2}\right| = |x-1| < 1$$

D.h. Reihe konvergiert f. $x \in (0,2)$

$x=0$: $\sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{2n-1}}_{\text{monoton fallende Nullfolge}}$ konvergiert (Leibniz)

313. Fortsetzung:

$x=2$: $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

$$\frac{1}{2n-1} > \frac{1}{2n}$$

$\sum_{n=1}^{\infty} \frac{1}{2n}$ ist Minorante

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \lim \left(\sum_{n=1}^{k} \frac{1}{2k} \right) = \lim \left(\frac{1}{2} \sum_{n=1}^{k} \frac{1}{k} \right) = \frac{1}{2} \cdot +\infty = +\infty$$

D.h. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ ist divergent

D.h. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2n-1}$ ist konvergent für $x \in [0,2)$.

---

314. $\sum_{n=0}^{\infty} \underbrace{\frac{x}{(1+x^2)^n}}_{a_n}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{x}{(1+x^2)^{n+1}}}{\frac{x}{(1+x^2)^n}} \right| = \left| \frac{1}{1+x^2} \right| \longrightarrow \left| \frac{1}{1+x^2} \right|$$

$$\left| \frac{1}{1+x^2} \right| < 1$$

$1 < |1+x^2| = 1+x^2$ gilt f.a. $x \neq 0, x \in \mathbb{R}$

D.h. Reihe konvergiert f.a. $x \neq 0$, $x \in \mathbb{R}$

$x=0$: $\sum_{n=0}^{\infty} \frac{0}{1^n} = \lim \sum_{n=0}^{k} \frac{0}{1^n} = \lim 0 = 0$

D.h. Reihe konvergiert f.a. $x \in \mathbb{R}$

315. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+1}(x+1)^n}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{(x+1)^{n+1}(n+1)}{n^2+2n+2}}{\frac{(x+1)^n n}{n^2+1}} \right| = \left| \frac{(x+1)(n+1)(n^2+1)}{n(n^2+2n+2)} \right|$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = |x+1| < 1$$

d.h. Reihe konvergiert in $(-2,0)$

$x=-2$: $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+1}}(-1)^n$ konvergiert (Leibniz)

(streng) monoton fallende Nullfolge

$x=0$: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} 1^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} > \frac{1}{2n} \quad \text{f.f.a.n}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = +\infty$, d.h. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n} = +\infty$

D.h. Reihe konvergiert in $[-2,0)$.

316. $\sum_{n=0}^{\infty} \underbrace{\frac{x^2}{(1+\sqrt[3]{x^2})^n}}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{x^2}{(1+\sqrt[3]{x^2})^{n+1}}}{\frac{x^2}{(1+\sqrt[3]{x^2})^n}} \right| = \left|\frac{1}{1+\sqrt[3]{x^2}}\right|$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{1+\sqrt[3]{x^2}}\right| < 1$$

$$1 < 1+\sqrt[3]{x^2}$$
$$0 < \sqrt[3]{x^2}$$
$$0 < x^2$$
$$\overline{x \neq 0}$$

$x=0$: $\sum_{n=0}^{\infty} \frac{0}{1^n} = 0$

D.h. Reihe konvergiert f.a. $x \in \mathbb{R}$

317. $\sum_{n\geq 0} \frac{x}{(1+x^2)^n} = \underbrace{\left(\frac{\sqrt[n]{x}}{1+x^2}\right)^n}_{q_n}$

$|q_n|<1$: $\frac{\sqrt[n]{x}}{1+x^2} < 1 \iff \sqrt[n]{x} < 1+x^2$

$\iff x < (1+x^2)^n$

Daher: $\sum_{n\geq 0} \frac{x}{\sqrt{1+x^2}} = \frac{1}{1-\frac{\sqrt[n]{x}}{1+x^2}} = \frac{1+x^2}{1+x^2-\sqrt[n]{x}}$

318. $$\sum_{n=0}^{\infty} \frac{x^2}{(1+\sqrt[3]{x^2})^n} = \sum_{n\geq 0} \underbrace{\left(\frac{\sqrt[n]{x^2}}{1+\sqrt[3]{x^2}}\right)^n}_{q_n}$$

$$|q_n|<1: \quad \frac{\sqrt[n]{x^2}}{1+\sqrt[3]{x^2}} < 1 \quad \Leftrightarrow \quad \sqrt[n]{x^2} < 1+\sqrt[3]{x^2}$$

gilt f.a. x

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+\sqrt[3]{x^2})^n} = \frac{1}{1-\frac{\sqrt[n]{x^2}}{1+\sqrt[3]{x^2}}} = \frac{1+\sqrt[3]{x^2}}{1+\sqrt[3]{x^2}-\sqrt[n]{x^2}}$$

323. $\cosh(x) = \frac{1}{2}\cdot(e^{x}+e^{-x})$

Potenzreihenentwicklung an $x_0 = 0$

$$\cosh(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

$$\cosh^{(k)}(x) = \begin{cases} \frac{1}{2}(e^{x}+e^{-x}) & \text{wenn } k=2n \\ \frac{1}{2}(e^{x}-e^{-x}) & \text{wenn } k=2n+1 \end{cases}$$

$$\cosh^{(k)}(x_0) = \cosh^{(k)}(0) = \begin{cases} 1 & \text{wenn } k=2n \\ 0 & \text{wenn } k=2n+1 \end{cases}$$

$$\cosh(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{k=0} \frac{\cosh^{(k)}(0)}{k!} x^k =$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

324. $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$

Potenzreihenentwicklung an $x_0 = 0$

$$\sinh^{(k)}(0) = \begin{cases} 0 & \text{wenn } k = 2n \\ 1 & \text{wenn } k = 2n+1 \end{cases}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

ansonsten wie 323

325. $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

LS: $\cosh(x+y) = \frac{1}{2}(e^{x+y} + e^{-x-y})$

RS: $\cosh(x)\cosh(y) + \sinh(x)\sinh(y) =$

$$= \frac{1}{2}(e^{x}+e^{-x})\frac{1}{2}(e^{y}+e^{-y}) + \frac{1}{2}(e^{x}-e^{-x})\frac{1}{2}(e^{y}-e^{-y}) =$$

$$= \frac{1}{4}(e^{x+y} \cancel{+e^{-x+y}} \cancel{+e^{x-y}} + e^{-x-y} + e^{x+y} \cancel{-e^{-x+y}} \cancel{-e^{x-y}} + e^{-x-y}) =$$

$$= \frac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \frac{1}{2}(e^{x+y} + e^{-x-y})$$

LS = RS

326. $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

LS: $\sinh(x+y) = \frac{1}{2}(e^{x+y} - e^{-x-y})$

RS: $\sinh(x)\cosh(y) + \cosh(x)\sinh(y) =$

$$= \frac{1}{2}(e^{x}-e^{-x})\frac{1}{2}(e^{y}+e^{-y}) + \frac{1}{2}(e^{x}+e^{-x})\frac{1}{2}(e^{y}-e^{-y}) =$$

$$= \frac{1}{4}(e^{x+y} \cancel{-e^{-x+y}} \cancel{+e^{x-y}} - e^{-x-y} + e^{x+y} \cancel{+e^{-x+y}} \cancel{-e^{x-y}} - e^{-x-y}) =$$

$$= \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \frac{1}{2}(e^{x+y} - e^{-x-y})$$

LS = RS

327. $$f(x) = (x^2+1)\underbrace{\sin x}_{g(x)}$$

$$g(x) = \sin x = \sum_{k=0}^{\infty} a_k (x-x_0)^k \quad ; \quad x_0 = 0$$

$$g(x) = \sum_{k=0}^{\infty} a_k x^k \qquad a_k = \frac{g^{(k)}(0)}{k!}$$

$$\sin^{(k)}(x) = \begin{cases} \sin x & \text{wenn } k = 4n \\ \cos x & \text{wenn } k = 4n+1 \\ -\sin x & \text{wenn } k = 4n+2 \\ -\cos x & \text{wenn } k = 4n+3 \end{cases}$$

$$\sin^{(k)}(0) = \begin{cases} 0 & \text{wenn } k = 2n \\ 1 & \text{wenn } k = 4n+1 \\ -1 & \text{wenn } k = 4n+3 \end{cases}$$

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \ldots = \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!}$$

$$f(x) = (x^2+1)\, g(x) = (x^2+1) \cdot \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!} =$$

$$= x + x^3 - \frac{x^3}{3!} - \frac{x^5}{3!} + \frac{x^5}{5!} + \frac{x^7}{5!} \mp \ldots =$$

$$= x - \left(\frac{x^3}{3!} - x^3\right) + \left(\frac{x^5}{5!} - \frac{x^5}{3!}\right) - \left(\frac{x^7}{7!} - \frac{x^7}{5!}\right) \pm \ldots$$

328. $f(x) = (1-x^2)\underbrace{\cos x}_{g(x)}$

$$g(x) = \cos x = \sum_{n\geq 0} a_n (x-x_0)^n = \qquad [x_0 = 0]$$

$$= \sum_{n \geq 0} a_n x^n \qquad a_n = \frac{g^{(n)}(x_0)}{n!} = \frac{g^{(n)}(0)}{n!}$$

$$\cos^{(n)}(0) = \begin{cases} 1 & n = 4k \\ 0 & \cancel{n=4k+1}\ n = 2k+1 \\ -1 & n = 4k+2 \end{cases}$$

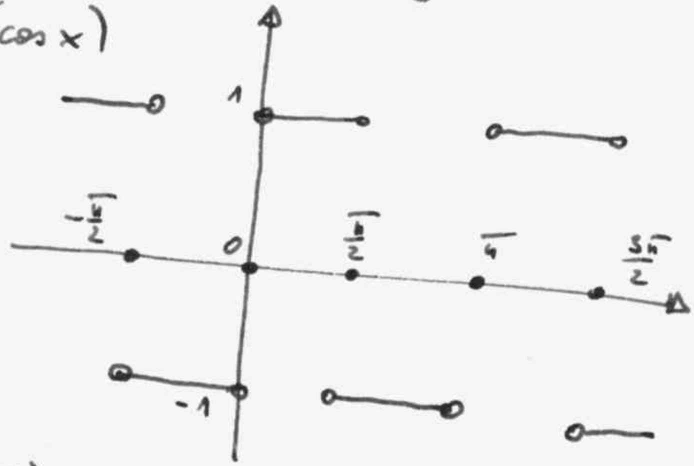
$$\cancel{\cos(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \ldots$$

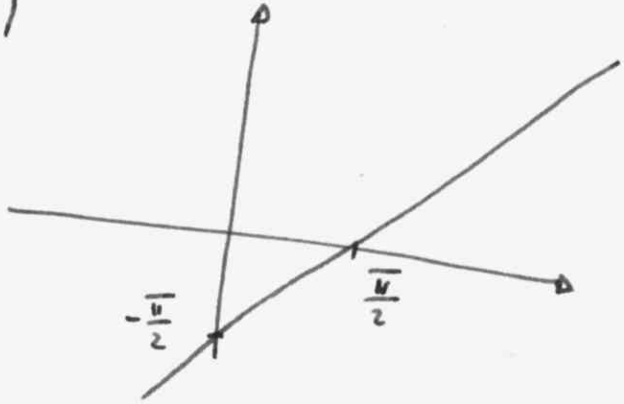
$$f(x) = (1-x^2)\cos x = 1 - \left(x^2 + \frac{x^2}{2!}\right) + \left(\frac{x^4}{2!} + \frac{x^4}{4!}\right) \cancel{\#} \pm \ldots$$

331. $f(x) = \left(x - \frac{\pi}{2}\right) \operatorname{sgn}(\cos x)$

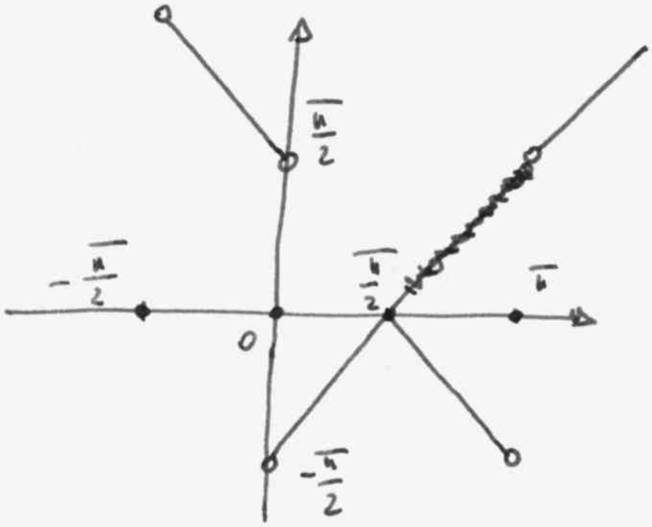
$\operatorname{sgn}(\cos x)$



$\left(x - \frac{\pi}{2}\right)$

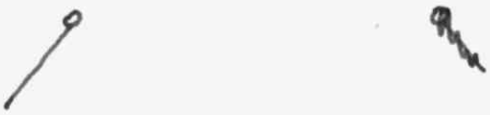


$\left(x - \frac{\pi}{2}\right) \operatorname{sgn}(\cos x)$

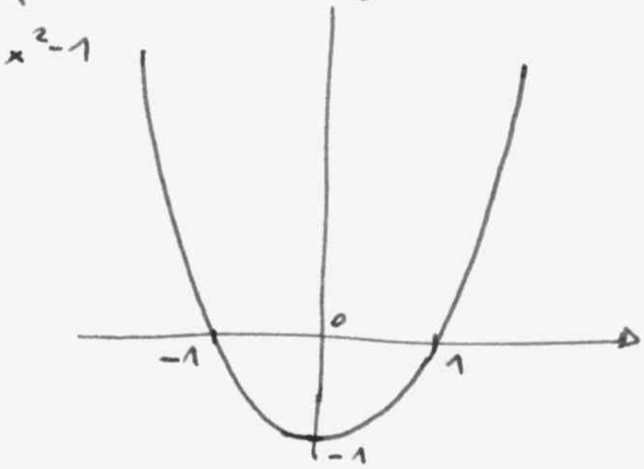


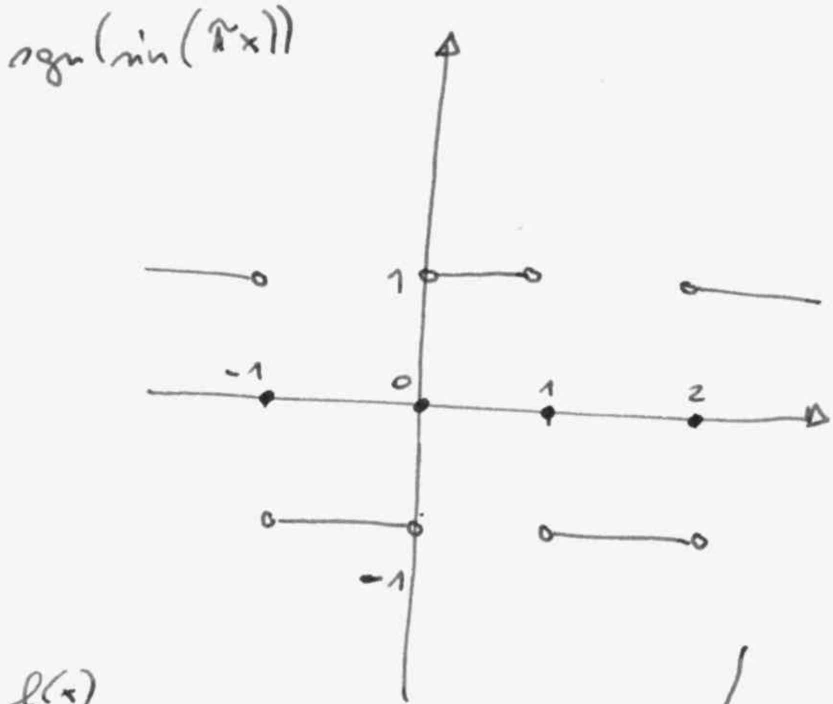
stetig in

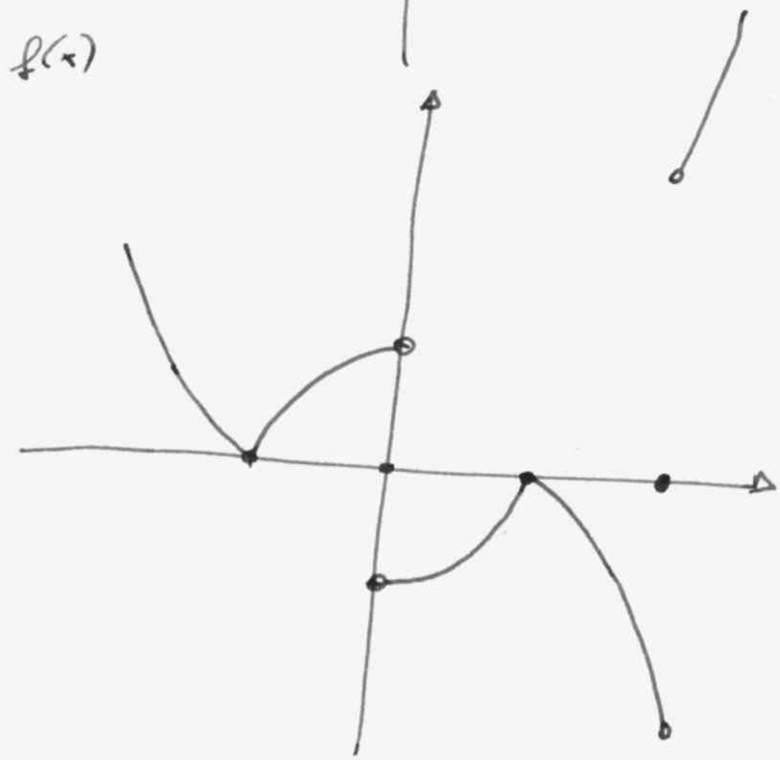
$\mathbb{R} \setminus \{ \frac{n\pi}{2} \mid n \in \mathbb{Z}, n \neq 1 \}$



332. $f(x) = (x^2-1)\,\mathrm{sgn}(\sin(\pi x))$

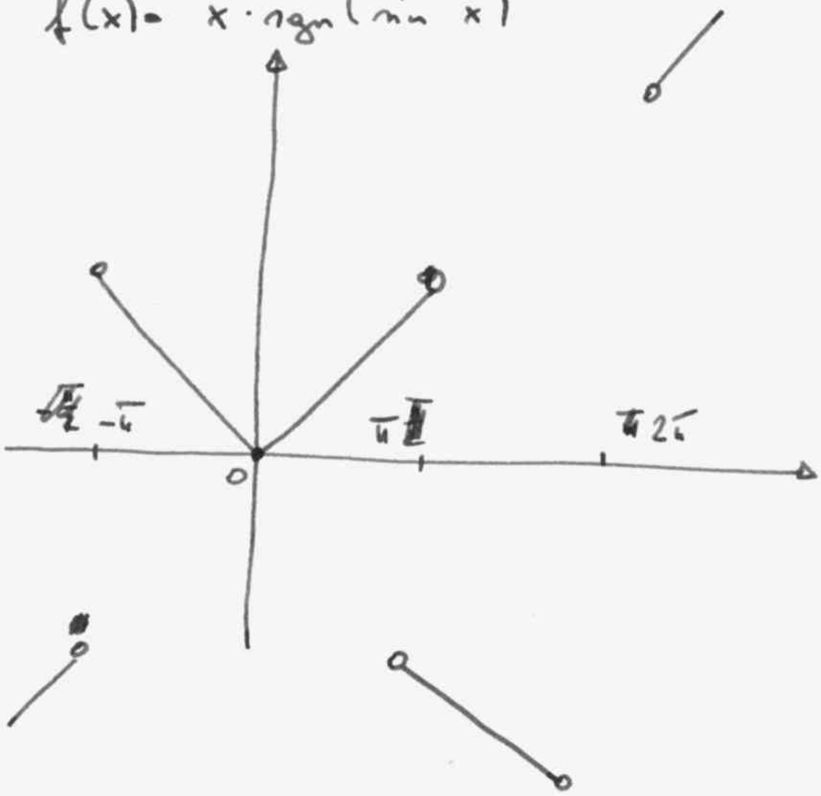






stetig für $\mathbb{R}\setminus(\mathbb{Z}\setminus\{1,-1\})$

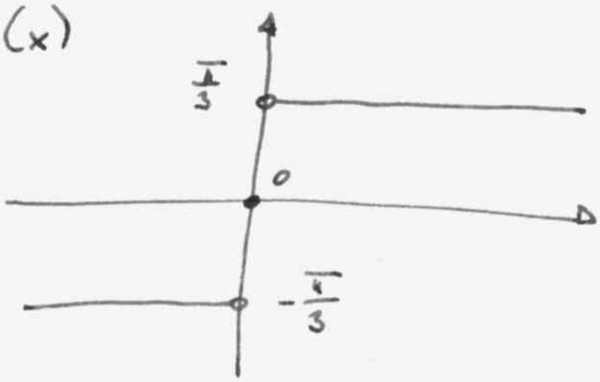
333. $f(x) = x \cdot \mathrm{sgn}(\sin x)$



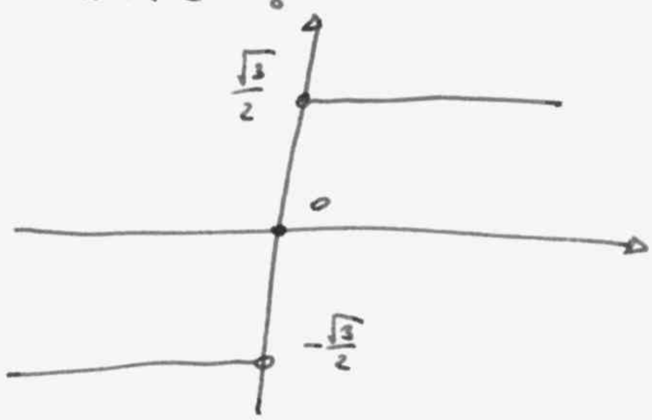
stetig f.a. ~~$x \in \mathbb{R} \setminus (k\mathbb{Z} \setminus \{0\})$~~
$x \in \mathbb{R} \setminus \{ n\pi \mid n \in \mathbb{Z}, n \neq 0 \}$

334. $f(x) = x \cdot \sin\left(\frac{\pi}{3}\,\mathrm{sgn}(x)\right)$

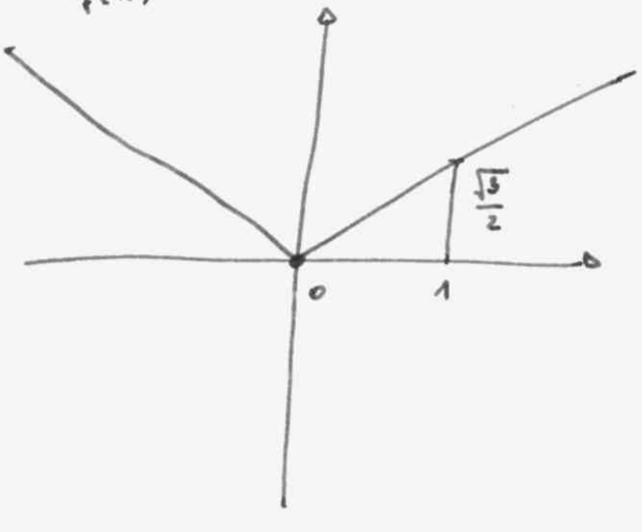
$\frac{\pi}{3} \cdot \mathrm{sgn}(x)$



$\sin\left(\frac{\pi}{3} \cdot \mathrm{sgn}(x)\right)$



$f(x)$



stetig in $\mathbb{R}$

335. $f(x,y) = \frac{|y|}{|x|^3 + |y|}$ für $(x,y) \neq (0,0)$; $f(0,0) = 1$

$\lim_{t \to 0} f(\alpha t, \beta t)$:

1, $\alpha = \beta = 0$: $\lim_{t \to 0} f(0,0) = 1$

2, $\alpha = 0$; $\beta \neq 0$: $\lim_{t \to 0} f(0, \beta t) = 1$

3, $\alpha \neq 0$; $\beta = 0$: $\lim_{t \to 0} f(\alpha t, 0) = 0$

4, $\alpha \neq 0$; $\beta \neq 0$:

$$\lim_{t \to 0} f(\alpha t, \beta t) = \lim_{t \to 0} \frac{|\beta t|}{|\alpha t|^3 + |\beta t|} = \lim_{t \to 0} \frac{|\beta|}{|\alpha|^3 |t|^2 + |\beta|} =$$

$$= \frac{|\beta|}{|\beta|} = 1$$

Wegen Fall 3, ist $f(x,y)$ an $(0,0)$ nicht stetig.

336. $f(x,y) = \frac{2y^2}{|x| + y^2}$ für $(x,y) \neq (0,0)$; $f(0,0) = 0$

$\lim_{t \to 0} f(\alpha t, \beta t)$:

1, $\alpha = \beta = 0$: $\lim_{t \to 0} f(0,0) = 0$

2, $\alpha = 0$; $\beta \neq 0$: $\lim_{t \to 0} f(0, \beta t) = 2$

3, $\alpha \neq 0$; $\beta = 0$: $\lim_{t \to 0} f(\alpha t, 0) = 0$

4, $\alpha \neq 0$, $\beta \neq 0$:

$$\lim_{t \to 0} f(\alpha t, \beta t) = \lim_{t \to 0} \frac{2\beta^2 t^2}{|\alpha t| + \beta^2 t^2} = \lim_{t \to 0} \frac{2\beta^2 |t|}{|\alpha| + \beta^2 |t|} = 0$$

Wegen Fall 2, ist $f(x,y)$ an $(0,0)$ nicht stetig.

337. $$f(x,y) = \frac{x\cos\frac{1}{x} + y\sin y}{2x-y} \qquad 0 \neq y \neq 2x$$

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{y\sin y}{-y} = \lim_{y\to 0} -\sin y = 0$$

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{x\cos\frac{1}{x}}{2x} = \lim_{x\to 0} \frac{\cos\frac{1}{x}}{2} \text{ existiert nicht}$$

Daher existiert auch $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ nicht.

338. $$f(x,y) = \frac{x + y\cos\frac{1}{y}}{x+y} \qquad 0 \neq y \neq -x$$

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{y\cos\frac{1}{y}}{y} = \lim_{y\to 0} \cos\frac{1}{y} \text{ existiert nicht}$$

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{x}{x} = 1$$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existiert nicht

339. $f(x,y) = \frac{xy}{|x|+|y|}$ für $(x,y) \neq 0,0$

$f(0,0) = 0$

Behauptung: $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

$$\lim_{(x,y)\to(0,0)} \left| \frac{xy}{|x|+|y|} - 0 \right| = 0$$

$$\left| \frac{xy}{|x|+|y|} \right| = \frac{|xy|}{|x|+|y|} \leq \frac{|xy|}{2\sqrt{|xy|}} = \frac{\sqrt{|xy|}}{2}$$

$$0 \leq \frac{|xy|}{|x|+|y|} \leq \frac{\sqrt{|xy|}}{2} \quad \text{f.a. } x,y$$

$$\lim_{(x,y)\to(0,0)} 0 = 0 \qquad \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{2} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$, also stetig

340. $f(x,y) = \frac{xy^2 + x^2y}{x^2+y^2}$ für $(x,y) \neq (0,0)$

$f(0,0) = 0$

~~Hilfsrechnung~~

$$f(x,y) = \frac{xy(x+y)}{x^2+y^2} \leq \frac{xy(x+y)}{2\sqrt{x^2y^2}} = \frac{x+y}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{2} = 0$$

$$0 \leq \left| \frac{xy^2+x^2y}{x^2+y^2} \right| \quad \text{f.a. } x,y$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy^2+x^2y}{x^2+y^2} = 0,$$ also stetig