1, $A \cup (B \cup C) = (A \cup B) \cup C$

$\max(x_A, \max(x_B, x_C)) = \max(\max(x_A, x_B), x_C))$

$\max(x_A(x), \max(x_B(x), x_C(x))) =$

$= \max(\max(x_A(x), x_B(x)), x_C(x)))$

LS: $\max(x_A(x), x_B(x), x_C(x))$

RS: $\max(x_A(x), x_B(x), x_C(x))$ f.a. $x \in M$

$x_A(x) + x_B(x) + x_C(x) - x_B(x) \cdot x_C(x) - x_A(x) \cdot (x_B(x) + x_C(x) -$
$- x_B(x) \cdot x_C(x)) =$

$= x_A(x) + x_B(x) - x_A(x) \cdot x_B(x) - x_C(x) \cdot (x_A(x) + x_B(x) - x_A(x) \cdot x_B(x))$

LS: $x_A(x) + x_B(x) + x_C(x) - x_B(x) \cdot x_C(x) - x_A(x) \cdot x_B(x) - x_A(x) \cdot x_C(x) +$
$+ x_A(x) \cdot x_B(x) \cdot x_C(x)$

RS: $x_A(x) + x_B(x) - x_A(x) \cdot x_B(x) - x_A(x) \cdot x_C(x) - x_B(x) \cdot x_C(x) +$
$+ x_A(x) \cdot x_B(x) \cdot x_C(x)$

$\cancel{a + b - ab - c \cdot (a + b - ab)}$

$a + b - ab + c - (a + b - ab) \cdot c$

$a + b - ab + c - ac - bc + abc$
$a + b + c - bc - ab - ac + abc$] =

② $A \cup (A \cap B) = A$

$\max(\chi_A(x), \min(\chi_A(x), \chi_B(x))) = \chi_A(x)$

$\chi_A(x) + \chi_A(x) \cdot \chi_B(x) - \chi_A^2(x) \cdot \chi_B(x) = \chi_A(x)$

LS: $\chi_A(x)$ RS: $\chi_A(x)$

---

③ $\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$

$a \cdot (b + c - bc) = ab + ac - abc$

LS: $ab + ac - abc$ RS: $ab + ac - abc$

④ $(A \setminus B) \setminus C = A \setminus (B \setminus C)$

$a - ab - (a - ab)c = a - a(b - bc)$

LS: $a - ab - ac + abc$

RS: $a - ab + abc$

$LS \neq RS$

$A = C = \{1\}$ $B = \{0\}$

⑤ $M \setminus (A \cap B) = (M \setminus A) \cup (M \setminus B)$

$\cancel{1 - ab = (1 - a)(1 - b)}$

$\cancel{1 - ab = 1 - a - b + ab}$

$1 - ab = (1 - a) + (1 - b) - (1 - a)(1 - b)$

RS: $\cancel{1} \cancel{- a} + \cancel{1} \cancel{- b} - \cancel{1} \cancel{+ a} \cancel{+ b} - ab$

(6) $M \setminus (A \Delta B) = (M \setminus A) \Delta (M \setminus B)$

$m - m(a + b - 2ab) = m - ma + m - mb - 2((m - ma)(m - mb))$

LS: $m - ma - mb + 2abm$

RS: $\cancel{m} - ma + \cancel{m} - mb - 2m + 2ma + 2mb - 2mab =$

$= \cancel{m}\ \cancel{2m} + ma + mb - 2abm$

(7) $A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

LS: $a + b - 2ab$

MS: $(a - ab) + (b - ba) - (a - ab)(b - ba)$

RS: $(a + b - ab) - (a + b - ab) \cdot ab$

MS: $a - ab + b - ab - ab + ab + ab - ab = a + b - 2ab$

RS: $a + b - ab - ab - ab + ab = a + b - 2ab$

(8) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

$a \cdot (b + c - 2bc) = ab + ac - 2abc$

LS: $ab + ac - 2abc = RS$

(9) $A \Delta (B \cap C) = (A \Delta B) \cap (A \overset{\cap}{\Delta} C)$

LS: $a + bc - 2abc$

RS: $(a + b - 2ab)(a + c - 2ac) =$

$= a + ab - 2ab + ac + bc - \cancel{2abc} - 2ac - \cancel{2abc} + \cancel{4abc} =$

$= a - ab - ac + bc$

(11) M … nichtleere, endliche Menge

$|M| = 0 \quad \Rightarrow \quad |P(M)| = 2^0 = 1$

$|M| = 1 \quad \Rightarrow \quad |P(M)| = 2^1 = 2$

$|M| = n \quad \Rightarrow \quad |P(M)| = 2^n$

$|M'| = n+1 \qquad |P(M)| = 2^{n+1}$

$M' = M \cup \{n+1\} \qquad P(\{n+1\}) = 2$

$|P(M')| = |P(M)| \cdot |P(\{n+1\})|$

$2^{n+1} = 2^n \cdot 2$

q.e.d.

12. i, $A \subseteq B$ $\quad x_A(x) \le x_B(x)$

ii, $A \cup B = B$ $\quad \max(x_A(x), x_B(x)) = x_B(x)$

iii, $A \cap B = A$ $\quad \min(x_A(x), x_B(x)) = x_A(x)$

14. $(A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)$

$A = \{1\}$

$B = \{2\}$

$A \times B = \{<1,2>\}$ $\qquad B \times A = \{<2,1>\}$

RS: $\ldots = \{<1,1>, <1,2>, <2,1>, <2,2>\}$

LS: $\ldots = \{<1,2>, <2,1>\}$

---

13. LS: $x_A(x) \cdot x_B(y) + x_B(x) \cdot x_A(y) - x_A(x) \cdot x_B(y) \cdot x_B(x) \cdot x_A(y)$

RS: $(x_A(x) + x_B(x) - x_A(x) \cdot x_B(x)) \cdot (x_A(y) + x_B(y) - x_A(y) x_B(y)) =$

$= x_A(x) \cdot x_A(y) + x_B(x) \cdot x_A(y) - x_A(x) \cdot x_B(x) \cdot x_A(y) +$

$+ x_A(x) \cdot x_B(y) - x_B(x) \cdot x_A(y) \cdot x_B(y) - x_A(x) \cdot x_B(x) \cdot x_A(y) -$

$- x_A(x) \cdot x_B(x) \cdot x_B(y) + x_A(x) \cdot x_A(y) \cdot x_B(x) \cdot x_B(y)$

13. LS: $x_A(x) \cdot x_B(y) \cdot x_A(y) \cdot x_B(y)$

RS: $x_A(x) \cdot x_B(x) \cdot x_A(y) \cdot x_B(y)$

15. $(A \times B) \cup (A \times C) = A \times (B \cup C)$

LS: $x_A(x) \cdot x_B(y) + x_A(x) \cdot x_C(y) - x_A(x) \cdot x_B(y) \cdot x_C(y)$

RS: $x_A(x) \cdot (x_B(y) + x_C(y) - x_B(y) \cdot x_C(y))$

LS = RS ✓

(16) $(A \times B) \cap (A \times C) = A \times (B \cap C)$

$$x_A(x) \cdot x_B(y) \cdot x_C(y) = x_A(x) \cdot x_B(y) \cdot x_C(y)$$

(18) R: mRm

S: $mRn \Leftrightarrow nRm$

weil $m = n \Leftrightarrow n = m$

$(m+n) \bmod 2 = (n+m) \bmod 2$

T: 1, $m = n$; $n = p \Rightarrow m = p$ ✓

2, $(m+n) \bmod 2 = 1$

$(n+p) \bmod 2 = 1 \Rightarrow (m+p) \bmod 2 = 0$

$\Rightarrow$ nicht transitiv

(19) mRn

$ggT(m,n) = 1$

R: $m \not R m$, weil $ggT(m,m) = m$

(21) ~~siehe 18. aber transitiv~~

$R = \{<m,n> \in A \times A \mid (m+n) \bmod 2 = 0\}$ $A = \{\ldots\}$

R: mRm

S: $mRn \Leftrightarrow nRm$

T: $(m+n) \bmod 2 = 0$

$(n+p) \bmod 2 = 0 \Rightarrow (m+p) \bmod 2 = 0$

(22) siehe 19

(23) $ARB \Leftrightarrow A \subseteq B \mid R = \{<a,b> \in A \times B \mid A \subseteq B;\ A, B \in P(M)\}$

R: ARA $A \subseteq A$

Sd: $A \subseteq B$; $B \subseteq A \Rightarrow A = B$

T: $A \subseteq B$; $B \subseteq C \Rightarrow A \subseteq C$ $C \in P(M)$

$\Rightarrow$ Äquivalenzrelation

(24.) $ARB \Leftrightarrow A \Delta B = \emptyset$

$R = \{\langle A, B\rangle \in \underset{}{P(M)\times P(M)} \mid A \Delta B = \emptyset;\ A, B \in P(M)\}$

$A \Delta B = \emptyset \Leftrightarrow A = B$

$\Rightarrow$ Äquivalenzrelation

(25.) $ARB \Leftrightarrow A \Delta B = A \Leftrightarrow B = \emptyset \Rightarrow R = \emptyset$

$R = \{\langle a, b\rangle \in P(M)\times P(M) \mid A \Delta B = A;\ A, B \in P(M)\} = \emptyset$

R: $A \Delta A = A$ falsch $\quad ARA$

S: $A \Delta B = A \Leftrightarrow B \Delta A = B$ falsch

$ARB \Leftrightarrow BRA$

(26) $R = \{\langle x, y\rangle \in A\times A \mid f(x) = f(y);\ f: A \to B\}$

R: $f(x) = f(x)$

S: $f(x) = f(y) \Leftrightarrow f(y) = f(x)$

T: $f(x) = f(y);\ f(y) = f(z) \Rightarrow f(x) = f(z)$

$\underline{z \in A}$

$\to$ Äquivalenzrelation

(27) R_1, R_2 HO $\Rightarrow R_1 \cap R_2$ HO ?

R: $\langle x, x\rangle \in R_1,\ \langle x, x\rangle \in R_2$

$\Rightarrow \langle x, x\rangle \in R_1 \cap R_2 \Rightarrow \underline{R}$

Ad: $\langle x, y\rangle \in R_1,\ \langle y, x\rangle \in R_1 \Leftrightarrow x = y$

$\langle x, y\rangle \in R_2,\ \langle y, x\rangle \in R_2 \Leftrightarrow x = y$

$\langle x, y\rangle \in R_1 \cap R_2,\ \langle y, x\rangle \in R_1 \cap R_2 \Leftrightarrow x = y$

T: $\langle x, y\rangle \in R_1,\ \langle y, z\rangle \in R_1 \Rightarrow \langle x, z\rangle \in R_1$

$\langle x, y\rangle \in R_2,\ \langle y, z\rangle \in R_2 \Rightarrow \langle x, z\rangle \in R_2$

$\langle x, y\rangle \in R_1 \cap R_2,\ \langle y, z\rangle \in R_1 \cap R_2 \Rightarrow \langle x, z\rangle \in R_1 \cap R_2$

(28.) R_1, R_2 ÄR $\Rightarrow$ $R_1 \cap R_2$ ÄR

R: ~~[illegible]~~ siehe 27.

T: siehe 27.

S: $\langle x, y \rangle \in R_1 \iff \langle y, x \rangle \in R_1$

$\langle x, y \rangle \in R_2 \iff \langle y, x \rangle \in R_2$

$\iff \langle x, y \rangle \in R_1 \cap R_2 \iff \langle y, x \rangle \in R_1 \cap R_2$

(29.) $mR_n \iff m^2 = n^2$

$R = \{ \langle m, n \rangle \in \mathbb{Z} \times \mathbb{Z} \mid m^2 = n^2 \}$

R: $m^2 = m^2$

S: $m^2 = n^2 \iff n^2 = m^2$; Id nicht, da S

T: $m^2 = n^2$; $n^2 = p^2 \Rightarrow m^2 = p^2$ $p \in \mathbb{Z}$

(30.) siehe 29.

(31) $mR_n \iff m = n^2$

$R = \{ \langle m, n \rangle \in \mathbb{Z} \times \mathbb{Z} \mid m = n^2 \}$

R: $m = m^2$ nein

S: $m = n^2 \iff n = m^2$ nein, ~~jedoch~~

Id: nein

T: $m = n^2, n = p^2 \Rightarrow m = p^2$ nein

32. Induktionsvoraussetzung:

$$\sum_{j=2}^{n} j(j-1) = \frac{(n-1)n(n+1)}{3}$$

Induktionsstart ($n=2$):

LS: $2(2-1) = 2$

RS: $\frac{1\cdot 2\cdot 3}{3} = 2$] =

Induktionsschritt:

$$\sum_{j=2}^{n+1} j(j-1) = \frac{n(n+1)(n+2)}{3}$$

LS: $\sum_{j=2}^{n+1} j(j-1) = \sum_{j=2}^{n} j(j-1) + (n+1)n =$

$= \frac{(n-1)n(n+1)}{3} + (n+1)n = \frac{n^3 \cancel{- n^2} \cancel{+ n^2} - 1n}{3} + \frac{3n^2+3n}{3} =$

$= \frac{n^3+3n^2+3n-1n}{3} = \frac{n^3+3n^2+2n}{3}$

RS: $\frac{n(n+1)(n+2)}{3} = \frac{n^3+2n^2+n^2+2n}{3} = \frac{n^3+3n^2+2n}{3}$

LS = RS

33. $\sum_{j=1}^{n} j(j+1) = \frac{n}{6}(2n^2+6n+4)$

$n=1$: LS: $1\cdot 2 = 2$

RS: $\frac{1}{6}\cdot(2+6+4) = 2$] =

$$\sum_{j=1}^{n+1} j(j+1) = \frac{n+1}{6}\left(2(n+1)^2+6(n+1)+4\right)$$

LS: $\sum_{j=1}^{n+1} j(j+1) = \sum_{j=1}^{n} j(j+1) + (n+1)(n+2) =$

$= \frac{n}{6}(2n^2+6n+4) + (n+1)(n+2) = \frac{n}{6}\left(2n^2+6n+4+\frac{6}{n}(n+1)(n+2)\right) =$

$= \cancel{\tfrac{1}{2}} \frac{1}{6}\left(2n^3+6n^2+4n+6\cancel{n}(n^2+3n+2)\right) =$

$= \frac{1}{6}\left(2n^3+6n^2+4n+6n^3+18n^2+12n\right) =$

$= \frac{1}{6}\left(\cancel{8n^3+24n^2+16n}\ 2n^3+12n^2+22n+12\right)$

RS: $\frac{n+1}{6}\left(2(n+1)^2+6(n+1)+4\right) =$

$= \frac{1}{6}(n+1)\left(2(n^2+2n+1)+6n+10\right) =$

$= \frac{1}{6}(n+1)(2n^2+4n+2+6n+10) = \frac{1}{6}(n+1)(2n^2+10n+12) =$

$= \frac{1}{6}(2n^3+10n^2+12n+2n^2+10n+12) = \frac{1}{6}(2n^3+12n^2+22n+12)$

---

34 $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \quad n \geq 1$

$n=1$: LS: $\frac{1}{2}$ RS: $\frac{1}{2}$

$\sum_{j=1}^{n+1} \frac{1}{j(j+1)} = \frac{n+1}{n+2}$

LS: $\sum_{j=1}^{n+1} \frac{1}{j(j+1)} = \sum_{j=1}^{n} \frac{1}{j(j+1)} + \frac{1}{(n+1)(n+2)} =$

$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$

LS = RS

(35) $$\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$$

$n=2:$ LS: $\frac{1}{2}$ RS: $\frac{1}{2}$

$$\sum_{j=2}^{n+1} \frac{1}{j(j-1)} = \frac{n}{n+1}$$

LS: $$\sum_{j=2}^{n+1} \frac{1}{j(j-1)} = \sum_{j=2}^{n} \frac{1}{j(j-1)} + \frac{1}{(n+1)n} = \frac{n-1}{n} + \frac{1}{(n+1)n} =$$

$$= \frac{n^2-1+1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1} = RS$$

(36) $$\sum_{j=0}^{n} j \cdot 2^j = 2^{n+1}(n-1)+2$$

$n=0:$ LS: $0 \cdot 2^0 = 0$
RS: $2^1(-1)+2 = 0$ $\Big] =$

$$\sum_{j=0}^{n+1} j \cdot 2^{j} = 2^{n+2} \cdot n+2$$

LS: $$\sum_{j=0}^{n+1} j \cdot 2^j = \sum_{j=0}^{n} j \cdot 2^j + (n+1) \cdot 2^{n+1} =$$

$$= 2^{n+1}(n-1)+2 + (n+1) \cdot 2^{n+1} =$$

$$= 2^{n+1}(n-1+n+1)+2 = 2^{n+1} \cdot 2n+2 = 2^{n+2} \cdot n+2$$

(37.) $\sum_{j=1}^{n} j \cdot 3^{j-1} = \frac{3^n(2n-1)+1}{4}$

$n=1$: LS: $1 \cdot 3^0 = 1$

RS: $\frac{3(1)+1}{4} = 1$

LS: $\sum_{j=1}^{n+1} j \cdot 3^{j-1} = \sum_{j=1}^{n} j \cdot 3^{j-1} + (n+1)3^n =$

$= \frac{3^n(2n-1)+1}{4} + (n+1) \cdot 3^n = \frac{3^n((2n-1)+4(n+1))+1}{4} =$

$= \frac{3^n(2n-1+4n+4)+1}{4} = \frac{3^n(6n+3)+1}{4}$

RS: $\frac{3^{n+1}(2n+1)+1}{4} = \frac{3^n(6n+3)+1}{4}$ LS = RS

38. $\sum_{k=1}^{n} k\cdot 5^k = \frac{5}{16}\cdot\left(n\cdot 5^{n+1} - (n+1)\cdot 5^n + 1\right)$

$n=1$:

LS $1\cdot 5 = 5$

RS $\frac{5}{16}\cdot\left(1\cdot 5^2 - 2\cdot 5 + 1\right) = \frac{5}{16}\cdot 16 = 5$ ✓

$$\sum_{k=1}^{n+1} k\cdot 5^k = \frac{5}{16}\cdot\left(n\cdot 5^{n+2} - (n+1)\cdot 5^{n+1} + 1\right)$$

$$\sum_{k=1}^{n+1} k\cdot 5^k = \sum_{k=1}^{n} k\cdot 5^k + (n+1)\cdot 5^{n+1} =$$

$$= \frac{5}{16}\cdot\left(n\cdot 5^{n+1} - (n+1)\cdot 5^n + 1\right) + (n+1)\cdot 5^{n+1} =$$

$$= \frac{5}{16}\left(n\cdot 5^{n+1} - (n+1)\cdot 5^n + 1 + \frac{16}{5}(n+1)5^{n+1}\right) =$$

$$= \frac{5}{16}\left(n\cdot 5^{n+1} - (n+1)\cdot 5^n + 1 + 16(n+1)5^n\right) =$$

$$= \frac{5}{16}\left(n\cdot 5^{n+1} + 15(n+1)\cdot 5^n + 1\right) =$$

$$= \frac{5}{16}\left(n\cdot 5^{n+1} + 3(n+1)\cdot 5^{n+1} + 1\right) =$$

$$= \frac{5}{16}\left((n + 3n + 1)\cdot 5^{n+1} + 1\right) =$$

$$= \frac{5}{16}\left((4n+1)\cdot 5^{n+1} + 1\right)$$

RS: $\frac{5}{16}\left(n \cdot 5^{n+2} - (n+1) \cdot 5^{n+1} + 1\right) =$

$= \frac{5}{16}\left(5n \cdot 5^{n+1} - (n+1) \cdot 5^{n+1} + 1\right) =$

$= \frac{5}{16}\left((5n - n - 1)\, 5^{n+1} + 1\right) =$

$= \frac{5}{16}\left((4n - 1)\, 5^{n+1} + 1\right)$

(39) $$\sum_{l=1}^{n} \frac{l}{3^l} = \frac{3}{4} - \frac{2n+3}{4\cdot 3^n}$$

$n=0$: LS: 0 RS: $\frac{3}{4} - \frac{3}{4} = 0$

$n=1$: LS: $\frac{1}{3}$ RS: $\frac{3}{4} - \frac{5}{12} = \frac{9}{12} - \frac{5}{12} = \frac{4}{12} = \frac{1}{3}$

$$\sum_{l=1}^{n+1} \frac{l}{3^l} = \frac{3}{4} - \frac{2(n+1)+3}{4\cdot 3^{n+1}}$$

LS: $$\sum_{l=1}^{n+1} \frac{l}{3^l} = \sum_{l=1}^{n} \frac{l}{3^l} + \frac{n+1}{3^{n+1}} =$$

$$= \frac{3}{4} - \frac{2n+3}{4\cdot 3^n} + \frac{n+1}{3^{n+1}} = \frac{3}{4} - \frac{6n+9}{4\cdot 3^{n+1}} + \frac{4n+4}{4\cdot 3^{n+1}} =$$

$$= \frac{3}{4} - \frac{2n+5}{4\cdot 3^{n+1}} = \frac{3}{4} - \frac{2(n+1)+3}{4\cdot 3^{n+1}}$$

LS=RS

(40) $\mathbb{N}\ \mathbb{Z}\ \mathbb{Q}\ \mathbb{R}\ \mathbb{C}$

$$F_0 = 0, \quad F_1 = 1; \quad F_{n+2} = F_{n+1} + F_n$$

$$F_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

$$F_0 = 0; \quad F_1 = 1 \quad \Rightarrow \quad F_2 = 1$$

$$F_0 = \frac{1}{\sqrt{5}} \cdot (1-1) = 0$$

$$F_1 = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5} - 1 + \sqrt{5}}{2} = \frac{1}{\sqrt{5}} \cdot \frac{2 \cdot \sqrt{5}}{2} = 1$$

$$F_2 = \frac{1}{\sqrt{5}} \cdot \left[\frac{1 + 2 \cdot \sqrt{5} + \sqrt{5} - 1 + 2 \cdot \sqrt{5} - \sqrt{5}}{4} \right] =$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{4 \cdot \sqrt{5}}{4} = \cancel{2}\ 1$$

$$F_{n+2} = F_{n+1} + F_n$$

$$\frac{1}{\sqrt{5}} \cdot \left(a^{n+2} - b^{n+2} \right) = \frac{1}{\sqrt{5}} \cdot \left(a^{n+1} - b^{n+1} \right) + \frac{1}{\sqrt{5}} \cdot \left(a^n - b^n \right)$$

RS: $\frac{1}{\sqrt{5}} \cdot \left(a^{n+1} - b^{n+1} + a^n - b^n \right) = \frac{1}{\sqrt{5}} \cdot \left(a^n (a+1) - b^n (b+1) \right) =$

$$= \frac{1}{\sqrt{5}} \cdot \left(a^n \left(\frac{1+\sqrt{5}}{2} + 1 \right) - b^n \left(\frac{1-\sqrt{5}}{2} + 1 \right) \right) =$$

$$= \frac{1}{\sqrt{5}} \cdot \left(a^n \left(\frac{1+\sqrt{5}+2}{2} \right) - b^n \left(\frac{1-\sqrt{5}+2}{2} \right) \right) =$$

$$= \frac{1}{\sqrt{5}} \cdot \left(a^n \left(\frac{3+\sqrt{5}}{2} \right) - b^n \left(\frac{3-\sqrt{5}}{2} \right) \right)$$

LS: $\frac{1}{\sqrt{5}}\cdot\left(a^n\cdot a^2 - b^n\cdot b^2\right) = \frac{1}{\sqrt{5}}\cdot\cancel{\left(a^n\cdot\frac{1+2\cdot\sqrt{5}+\sqrt{5}}{4} - b^n\cdot\frac{1-2\sqrt{5}+\sqrt{5}}{4}\right)} =$

$$\cancel{=\frac{1}{\sqrt{5}}\left(a^n\cdot\frac{1+3\cdot\sqrt{5}}{4} - b^n\cdot\frac{1-3\cdot\sqrt{5}}{4}\right)}$$

$$=\frac{1}{\sqrt{5}}\cdot\left(a^n\cdot\frac{1+2\cdot\sqrt{5}+5}{4} - b^n\cdot\frac{1-2\cdot\sqrt{5}+5}{4}\right) =$$

$$=\frac{1}{\sqrt{5}}\left(a^n\cdot\frac{3+\sqrt{5}}{2} - b^n\cdot\frac{3-\sqrt{5}}{2}\right)$$

---

(41) $L_0 = 2;\ L_1 = 1\ ;\ L_{n+2} = L_{n+1} + L_n \Rightarrow L_2 = 3$

$$L_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$L_0 = 1+1 = 2$

$L_1 = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = \frac{2}{2} = 1$

$L_2 = \frac{1+2\sqrt{5}+5+1-2\cdot\sqrt{5}+5}{4} = \frac{12}{4} = 3$

$$a^{n+2} + b^{n+2} = a^{n+1} + b^{n+1} + a^n + b^n$$

RS: $a^n(a+1) + b^n(b+1) = a^n\left(\frac{1+\sqrt{5}+2}{2}\right) + b^n\left(\frac{1-\sqrt{5}+2}{2}\right) =$

$$= a^n\left(\frac{3+\sqrt{5}}{2}\right) + b^n\left(\frac{3-\sqrt{5}}{2}\right)$$

LS: $a^n\cdot a^2 + b^n\cdot b^2 = a^n\cdot\left(\frac{1+2\cdot\sqrt{5}+5}{4}\right) + b^n\left(\frac{1-2\sqrt{5}+5}{4}\right) =$

$$= a^n\left(\frac{3+\sqrt{5}}{2}\right) + b^n\left(\frac{3-\sqrt{5}}{2}\right)$$

(42) $F_0 = 0$; $F_1 = 1$; $F_{n+2} = F_{n+1} + F_n$

$F_n < \left(\frac{7}{4}\right)^n$ $\qquad$ $F_{n+2} < \left(\frac{7}{4}\right)^{n+2}$

$n=0$: $\quad 0 < 1$ $\qquad$ $F_{n+1} + F_n < \left(\frac{7}{4}\right)^{n+2}$

$n=1$: $\quad 1 < \frac{7}{4}$

~~$n=2$~~

$F_{n+1} < \left(\frac{7}{4}\right)^{n+1}$

$F_n < \left(\frac{7}{4}\right)^{n}$

$F_{n+1} + F_n < \left(\frac{7}{4}\right)^{n+1} + \left(\frac{7}{4}\right)^{n}$

$F_{n+1} + F_n < \left(\frac{7}{4}\right)^{n} \cdot \frac{11}{4} < \left(\frac{7}{4}\right)^{n} \cdot \frac{49}{16}$

(43) $9n^3 - 3 \leq 8^n$

$n=0$: $\quad -3 \leq 1$

$n=1$: $\quad 9 - 3 \leq 8$

$n=2$: $\quad 72 - 3 \leq 64$

$n=3$: $\quad 243 - 3 \leq 512$

$9(n+1)^3 - 3 \leq 8^{n+1}$

LS: $\quad 9(n^3 + 3n^2 + 3n + 1) - 3$ ~~$\leq 8^{n+1}$~~ $=$

$9n^3 + 27n^2 + 27n + 6 \leq 8^{n+1}$

$64 \cdot 8^{n-1}$

$9n^3 + 27n^2 + 27n + 6 \leq 9 \cdot 8^{n-1} + 27 \cdot 8^{n-1} + 27 \cdot 8^{n-1} + 8^{n-1}$

$9 \cdot n^3 \leq 9 \cdot 8^{n-1}$ $\qquad$ $n^3 \leq 8^{n-1}$

$27n^2 \leq 27 \cdot 8^{n-1}$ $\qquad$ $n^2 \leq 8^{n-1}$

$27n \leq 27 \cdot 8^{n-1}$ $\qquad$ $n \leq 8^{n-1}$

$6 \leq 8^{n-1}$ $\qquad$ $6 \leq 8^{n-1}$

(43) $9n^3 - 3 \leq 8^n$ Annahme: $A(n)$ wahr

$n = 3$: $240 \leq 512$

$9(n+1)^3 - 3 \leq 8^{n+1}$

$9n^3 + 27n^2 + 27n + 9 - 3 \leq 8 \cdot 8^n$

$27n^2 + 27n + 9 \leq 7 \cdot 8^n$ Annahme: $A(n+1)$ wahr

$27(n+1)^2 + 27(n+1) + 9 \leq 7 \cdot 8^{n+1}$

$\underline{27n^2} + 54n + 27 + \underline{27n} + 27 + \underline{9} \leq \cancel{7 \cdot 8^n \cdot 8} \; 56 \cdot 8^n$

$54n + 54 \leq 49 \cdot 8^n$ Annahme: $A(n+2)$ wahr

$54(n+1) + 54 \leq 49 \cdot 8 \cdot 8^n$

$54n + 108 \leq 392 \cdot 8^n$

$54 \leq 343 \cdot 8^n$

$\frac{54}{343} \leq 8^n$

$L = \{n \in \mathbb{N} \mid n \neq 2\}$

$L = \{n \in \mathbb{N} \setminus \{2\}\}$

$L = \mathbb{N} \setminus \{2\}$???

---

(44)

(44) ~~$4n^2 \leq 2^n$~~

$$4n^2 \leq 2^n$$

$$4(n+1)^2 \leq 2^{n+1}$$

$$4n^2 + 8n + 4 \leq 2^n \cdot 2$$

$$8n + 4 \leq 2^n$$

~~$8n \leq 2^n$~~

$$8(n+1) + 4 \leq 2^n \cdot 2$$

$$8n + 12 \leq 2^n \cdot 2$$

$$8 \leq 2^n \quad \text{w.A. f.a. } \underline{n \geq 3}$$

$n=0$:	$0 \leq 1$
$n=1$:	$4 \leq 2$
$n=2$:	$16 \leq 4$
$n=3$:	$36 \leq 8$
$n=4$:	$64 \leq 16$
$n=5$:	$100 \leq 32$
$n=6$:	$144 \leq 64$
$n=7$:	$196 \leq 128$
$n=8$:	$256 \leq 256$
$n=9$:	$324 \leq 512$

=> Aussage wahr f.a. $n \geq 8$

(47.) $$\sum_{k=1}^{n} a_k b_k = a_n \sum_{k=1}^{n} b_k - \sum_{k=1}^{n-1} (a_{k+1} - a_k) \cdot \sum_{j=1}^{k} b_j$$

$n=1$: $$a_1 b_1 = a_1 b_1 - 0$$

$n=2$: $$a_2 b_2 + a_1 b_1 = a_2 \cdot (b_1 + b_2) - (a_2 - a_1) \cdot b_1$$

$$a_2 b_1 + a_2 b_2 - a_2 b_1 + a_1 b_1 = a_1 b_1 + a_2 b_2$$

$$\sum_{k=1}^{n+1} a_k b_k = a_{n+1} \sum_{k=1}^{n+1} b_k - \sum_{k=1}^{n} (a_{k+1} - a_k) \cdot \sum_{j=1}^{k} b_j$$

LS: $$\sum_{k=1}^{n+1} a_k b_k = \left(\sum_{k=1}^{n} a_k b_k\right) + a_{n+1} b_{n+1} =$$

$$= a_n \sum_{k=1}^{n} b_k - \sum_{k=1}^{n-1} (a_{k+1} - a_k) \cdot \sum_{j=1}^{k} b_j + a_{n+1} b_{n+1}$$

RS: $$a_{n+1}\left[\left(\sum_{k=1}^{n} b_k\right) + b_{n+1}\right] - \sum_{k=1}^{n-1} (a_{k+1} - a_k) \cdot \sum_{j=1}^{k} b_j - (a_{n+1} - a_n) \cdot \sum_{j=1}^{n} b_j =$$

$$= a_{n+1} \cdot b_{n+1} - \sum_{k=1}^{n-1} (a_{k+1} - a_k) \cdot \sum_{j=1}^{k} b_j + a_n \cdot \sum_{j=1}^{n} b_j$$

LS=RS

(50) $M = \{0, 1, 2\}$, $m \circ n = \min(m+n, 2)$

G1: $0 \leq \min(m+n, 2) \leq 2$ $\Rightarrow$ abgeschlossen

G2: $(m \circ n) \circ o = m \circ (n \circ o)$

~~$\min(\min(m+n, 2), 2) = \min(m$~~

$\min(\min(m+n, 2) + o, 2) = \min(m + \min(n+o, 2), 2)$

$\min(m+n+o, 2) = \min(m+n+o, 2)$

$\Rightarrow$ assoziativ

G3: $e = 0$

$m \circ e = \min(m+e, 2) = \min(m+0, 2) = m$ $\Rightarrow$ $\underline{e = 0}$

G4: $m \circ m' = m' \circ m = e$ $\Rightarrow$ kein m'

$\Rightarrow$ $\underline{\text{MONOID}}$

(51) $M = \{0, 1, 2, 3\}$; $m \circ n = \min(m \cdot n, 3)$

G1: ✓

G2: ✓

G3: $\underline{e = 1}$

G4: kein m' $\Rightarrow$ $\underline{\text{MONOID}}$

(52) $M = \{-2, 1, 0, 1, 2\}$; $m \circ n = mn$

G1: nein $\Rightarrow$ $\underline{\text{nix}}$

(53) $M = \{z \in \mathbb{C} \mid |z| = 2\}$ $z_1 \circ z_2 = \frac{z_1 \cdot z_2}{2}$

G1: abgeschlossen, weil $\left|\frac{z_1 \cdot z_2}{2}\right| = 2$

G2:

$$\frac{\frac{z_1 \cdot z_2}{2} \cdot z_3}{2} = \frac{z_1 \cdot \frac{z_2 \cdot z_3}{2}}{2}$$

$$\frac{z_1 \cdot z_2 \cdot z_3}{4} = \frac{z_1 \cdot z_2 \cdot z_3}{2}$$

G3: $\underline{e = 2 + 0 \cdot i}$ $\frac{z_1 \cdot e}{2} = z_1$ $\Rightarrow$ $e = \frac{z_1}{z_1} \cdot 2 + 0 \cdot i$

G4: $\frac{z_1 \cdot z_1'}{2} = e$ $z_1' = \frac{2e}{z_1} = \frac{4}{z_1}$

$\Rightarrow$ $\underline{\text{GRUPPE}}$

(54) $M = \{ z \in \mathbb{C} \mid |z| = 1 \}$ $\quad z_1 \circ z_2 = z_1 z_2$

G1: ✓ $\quad (|z_1 z_2| = |z_1| \cdot |z_2| = 1 \cdot 1 = 1)$

G2: $|(z_1 \cdot z_2) \cdot z_3| = |z_1 \cdot (z_2 \cdot z_3)|$

$\vdots$

$|z_1| \cdot |z_2| \cdot |z_3| = |z_1| \cdot |z_2| \cdot |z_3| \Rightarrow$ assoziativ

G3: $\underline{e = 1 + i \cdot 0}$

$z_1 \circ e = z_1$

$z_1 \cdot e = z_1 \Rightarrow \underline{e = 1 + i \cdot 0}$

G4: $z_1 \circ z_1' = e$

$z_1 \cdot z_1' = e$

$z_1' = \frac{e}{z_1} = \frac{1}{z_1} \Rightarrow$ GRUPPE

(55) $M = \{ z \in \mathbb{C} \mid |z| = 2 \}$; $z_1 \circ z_2 = z_1 \cdot z_2$

G1: nein $\Rightarrow$ nix

(57) $B, C \in M = P(A)$; $B, C \subseteq A$; $B \circ C = B \cup C$

G1: $B \cup C \subseteq A \Rightarrow B \cup C \in M$

G2: assoziativ

G3: $E = \emptyset$

G4: kein inverses Element

$\Rightarrow$ MONOID, wenn $A \neq \emptyset$

GRUPPE ~~keine algebr. Str.~~ GRUPPE, wenn $A = \emptyset$

(58) $B \circ C = B \cap C$

G1: $B \cap C \in M \Rightarrow$ abgeschlossen

G2: assoziativ:

$(\chi_B(x) \cdot \chi_C(x)) \cdot \chi_D(x) = \chi_B(x) \cdot (\chi_C(x) \cdot \chi_D(x))$

G3: $E = A$

G4: kein inverses Element, wenn $A \neq \emptyset$, sonst $B' = \emptyset$

MONOID, wenn $A \neq \emptyset$

GRUPPE, wenn $A = \emptyset$

(59) $B \circ C = B \triangle C$

G1: ✓

G2: $(B \triangle C) \triangle D = B \triangle (C \triangle D)$

$$(b + c - 2bc) + d - 2d(b + c - 2bc) = b + (c + d - 2cd) - -2b(c + d - 2cd)$$

$$b + c - 2bc + d - bd - cd - 2bcd = b + c + d - 2cd - bc - \cancel{bd} - \cancel{2bcd}$$

⋮

G3: $E = \emptyset$

G4: ~~nur für $A = \emptyset$~~ $A \triangle A' = E$ $A' = A$

~~=> MONOID/GRUPPE~~ => GRUPPE

(60) $B \circ C = B \setminus C$

G1: ✓

G2: nein (siehe 4) => GRUPPOID

$A = \emptyset$ -> GRUPPE ?

(61) $\langle M, \oplus_2, \cdot \rangle$ $M = \{0, 1\}$ $a \cdot b = 0$ f.a. $a, b \in M$

$\langle M, \oplus_2 \rangle$:

G1: abgeschlossen

G2: assoziativ

G3: $e = 0$

G4: $a' = a$ f.a. $a \in M$

G5: • kommutativ ($a \oplus_2 b = b \oplus_2 a$

=> ABELSCHE GRUPPE

$\langle$ ~~M~~ $M, \cdot \rangle$: G1: abgeschlossen

G2: assoziativ (Produkt immer 0)

G3: kein Einheitselement

=> HALBGRUPPE

DG: $a \cdot (b+c) = a \cdot b + a \cdot c$
$[\ (a+b) \cdot c = a \cdot c + b \cdot c\ ]$

$\Rightarrow$ RING

(62) $M = \{0,1,2\}$ $\quad a \cdot b = 1$ $\quad \langle M, \oplus_3, \cdot \rangle$

$\langle M, \oplus_3 \rangle$: G1: ✓
G2: ✓
G3: $e = 0$
G4: $0' = 0$; $1' = 2$; $2' = 1$ $\Rightarrow$ ✓
G5: ✓ $\Rightarrow$ ABELSCHE GRUPPE

$\langle M, \cdot \rangle$: G1: ✓
G2: ✓
G3: kein e $\Rightarrow$ HALBGRUPPE

DG ✓

RING

(63) $M = \mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ $\quad \langle M, +, \cdot \rangle$

$\langle M, + \rangle$: G1: $(a + b\sqrt{5}) + (c + d\sqrt{5}) =$ $\quad c, d \in \mathbb{Q}$

$= (a+c) + (b+d)\sqrt{5}$

$a + c \in \mathbb{Q}$, $b + d \in \mathbb{Q}$ $\Rightarrow$ abgeschlossen

G2:

⋮

$(a+c+e) + (b+d+f)\sqrt{5} = (a+c+e) + (b+d+f)\sqrt{5}$

G3: $e = 0 + 0 \cdot \sqrt{5}$
G4: $(a + b\sqrt{5})' = (-a) + (-b)\sqrt{5}$
G5: trivial $\Rightarrow$ ABELSCHE GRUPPE

$\langle M, \cdot \rangle$: G1: $(a + b\sqrt{5})(c + d\sqrt{5}) =$

$= ac + bc\sqrt{5} + ad\sqrt{5} + 5bd = \underbrace{(ac + 5bd)}_{\in \mathbb{Q}} + \underbrace{(bc + ad)}_{\in \mathbb{Q}}\sqrt{5}$

G2: ✓

G3: $e = 1 + 0 \cdot \sqrt{5}$

G4: $(a + b \cdot \sqrt{5}) \cdot (a' + b'\sqrt{5}) = 1 + 0 \cdot \sqrt{5}$

$$a' + b'\sqrt{5} = \frac{1 + 0 \cdot \sqrt{5}}{a + b \cdot \sqrt{5}}$$

$$a' + b'\sqrt{5} = \frac{(1 + 0 \cdot \sqrt{5}) \cdot (a - b\sqrt{5})}{a^2 - 5b^2}$$

$$a^2 = 5b^2$$

$$a = \pm\sqrt{5} \cdot b$$

G5: kommutativ

=> ABELSCHE GRUPPE

=> KÖRPER

64 wie 63, jedoch $\langle \mathbb{Q}[\sqrt{4}], \cdot \rangle$ nicht abelsche Gruppe, sondern nur kommutatives Monoid, also RING

65 wie 63

66 wie 64

67 $\langle M, \oplus_3, \odot_4 \rangle$ $M = \{0, 1, 2\}$

$\langle M, \oplus_3 \rangle$

G1: ✓

G2: ✓

G3: $e = 0$

G4: $0' = 0$

$1' = 2$

$2' = 1$

G5: ✓

ABELSCHE GRUPPE

$\langle M, \odot_4 \rangle$

G1: ✓

G2: ✓

G3: $e = 1$

G4: kein inverses Element

G5: ✓

ABELSCHES MONOID

=> RING

68 $M = \{0,1\}$

$\langle M, +, \odot_2 \rangle$

$0+0=0$

$0+1=1+0=1$

$1+1=1$

$\langle M, + \rangle$:

G1: ✓

G2: ✓

G3: $\underline{e=0}$

G4: ~~[illegible]~~ kein inverses El.

~~[illegible]~~ G5: ✓

ABELSCHES MONOID

-> HALBRING

$\langle M, \odot_2 \rangle$

G1: ✓

G2: ✓

G3: $e=1$

G4: ~~[illegible]~~ kein ~~inver~~ inverses Element

G5: ✓

ABELSCHES MONOID

(67.) $M = \{0, 1, 2\}$

$\langle M, \oplus_3, \odot_4 \rangle$

$\langle M, \oplus_3 \rangle$

G1: $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

G2: ✓

G3: $\underline{e = 0}$

G4: $0' = 0$; $1' = 2$; $2' = 1$

G5: ✓

ABELSCHE GRUPPE

$\Rightarrow$ RING

$\langle M, \odot_4 \rangle$

G1: ✓

G2: ✓

G3: $\underline{e = 1}$

G4: nein

G5: ✓

ABELSCHES MONOID

(68.) $M = \{0, 1\}$

$\langle M, +, \odot_2 \rangle$

$0 + 0 = 0$

$0 + 1 = 1 + 0 = 1$

$1 + 1 = 1$

$\langle M, + \rangle$

G1: ✓

G2: ✓

G3: $\underline{e = 0}$

G4: nein

G5: ✓

KOMMUTATIVES MONOID

$\Rightarrow$ HALBRING

$\langle M, \odot_2 \rangle$

G1: ✓ $(\mathbb{Z}_2 = \{0, 1\})$

G2: ✓

G3: $\underline{e = 1}$

G4: nein

G5: ✓

KOMMUTATIVES MONOID

(69) a) $8x \equiv 4 \pmod{16}$

$8x - 4 = 16 \cdot q_n \quad q \in \mathbb{Z}$

$\left(\frac{1}{2}x - \frac{1}{4} = q_n\right)$

$\frac{16q_n + 4}{8} = x$

$2q_n + \frac{1}{2} = x$

~~L = {x ∈ / 2q_n +~~

$x = \{2q_n + \frac{1}{2} \mid q_n \in \mathbb{Z}\}$

b) $8x \equiv 4 \pmod{15}$

$8x = 4 + 15q_n$

$x = \frac{4 + 15q_n}{8}$

$x = \{\frac{1}{2} + \frac{15q_n}{8} \mid q_n \in \mathbb{Z}\}$

(75) $K \subseteq \mathbb{C}$

$\mathbb{R} \subseteq K$

$1+3i \in K$

$\langle K, +, \cdot \rangle$ ist Körper

$1+3i \in K \Rightarrow 1+3i+(-1) \in K$

$\Rightarrow 3i \in K$

$\Rightarrow i \in K$

$a \in \mathbb{R} \Rightarrow a \cdot i \in K$

$b \in \mathbb{R} \Rightarrow b + a \cdot i \in K$

$\mathbb{C} = \{ b + a \cdot i \mid a, b \in \mathbb{R} \}$

$\{ b + a \cdot i \mid a, b \in \mathbb{R} \} \subseteq K$

$\mathbb{C} \subseteq K \Rightarrow \mathbb{C} = K$

(76) $1-i \in K$, sonst wie 75

$\Rightarrow 1 - i + (-1) \in K$

$-i \in K$

$i \in K$ weiter wie 75

(77) $\mathbb{R} \subset K \subset \mathbb{C}$; K ist Körper

$z \in \mathbb{C}$ $z = a + b \cdot i$, $a, b \in \mathbb{R}$, $z \notin \mathbb{R}$

es muss mind. 1 solches z in K enthalten sein (sonst $K = \mathbb{R}$)

wenn $z \in K$, $a + b \cdot i \in K$

$a + b \cdot i + (-a) \in K$

$b \cdot i \in K$

$b \cdot i \cdot \frac{1}{b} \in K \Rightarrow i \in K$

$\Rightarrow c, d \in \mathbb{R} \Rightarrow c + d \cdot i \in K$

$\Rightarrow K \supseteq \{ c + d \cdot i \mid c, d \in \mathbb{R} \}$

$\mathbb{C} = \{ c + d \cdot i \mid c, d \in \mathbb{R} \}$

$K \subset \mathbb{C}$; $\mathbb{C} \subseteq K$ $\Rightarrow$ Widerspruch

$\Rightarrow$ Es gibt keine solche Menge K.

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$\odot_5$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

$L = \{ \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}, \{\bar{1}\}, \{\bar{1}, \bar{4}\} \}$

79

$+_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$L = \{ \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}, \{\bar{0}\}, \{\bar{0}, \bar{2}\} \}$

80

(81.) $\mathbb{Z}_9$; $E(\mathbb{Z}_9) = ?$

$ggT(0,9) = ?$

$ggT(1,9) = 1$

$ggT(2,9) = 1$

$ggT(3,9) = 3$

$ggT(4,9) = 1$

$ggT(5,9) = 1$

$ggT(6,9) = 3$

$ggT(7,9) = 1$

$ggT(8,9) = 1$

$\Rightarrow E(\mathbb{Z}_9) = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$

(82.) $E(\mathbb{Z}_6) = \{\bar{1}, \bar{5}\}$ $\qquad E(\mathbb{Z}_3) = \{\bar{1}, \bar{2}\}$

$\Rightarrow$ isomorph : $E(\mathbb{Z}_3) \cong E(\mathbb{Z}_6)$

(83.) $(-a)\cdot b = -(a\cdot b)$

$0 = 0$

$((-a)+a)\cdot b = 0$

$(-a)\cdot b + a\cdot b = 0$

$(-a\cdot b) = -(a\cdot b)$

(84.)

$0 = 0$

$a\cdot((-b)+b) = 0$

$a\cdot(-b) + a\cdot b = 0$

$a\cdot(-b) = -(a\cdot b)$

(85.)

$0 = 0$

$0\cdot 0 = 0$

$((-a)+a)\cdot((-b)+b) = 0$

$(-a)\cdot(-b) + a\cdot(-b) + \underbrace{(-a)\cdot b + a\cdot b}_{=0 \text{ (siehe 83)}} = 0$

$(-a)\cdot(-b) + a\cdot(-b) = 0$ (siehe 84.)

$(-a)\cdot(-b) + (-(a\cdot b)) = 0$

$(-a)\cdot(-b) = a\cdot b$

(86) $\langle 1, 1 \rangle + 2\cancel{x} = \langle 1, 3 \rangle$

~~$M = \mathbb{Z}_2 \times \mathbb{Z}_4$~~

$\langle \mathbb{Z}_2, \oplus_2 \rangle, \quad \langle \mathbb{Z}_4, \oplus_4 \rangle$

$M = \langle \mathbb{Z}_2, \oplus_2 \rangle, \langle \mathbb{Z}_4, \oplus_4 \rangle$

$1 + 2x_1 = 1 \qquad x_1 \in \mathbb{Z}_2$

$1 + 2x_2 = 3 \qquad x_2 \in \mathbb{Z}_4$

$L = \{ \langle \bar{0}, \bar{1} \rangle, \langle \bar{0}, \bar{3} \rangle, \langle \bar{1}, \bar{3} \rangle, \langle \bar{1}, \bar{1} \rangle \}$

(87) $\mathbb{Z}_2$:

	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

$L_1 = \{ \{\bar{0}, \bar{1}\}, \{\bar{0}\} \}$

$\mathbb{Z}_4$:

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$L_2 = \{ \{\bar{0}\}, \{\bar{0}, \bar{2}\}, \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \}$

$L = L_1 \times L_2$

$|L| = 6$

(88) $\langle 1, 1 \rangle + 2z = \langle 2, 1 \rangle$

$L = \{ \langle 2, 0 \rangle, \langle 2, 1 \rangle \}$

(89) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ 0 \end{pmatrix}$; $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ 0 \end{pmatrix}$

$\langle \mathbb{R}^2, + \rangle$:

G1: ✓ $(x_1+y_1 \in \mathbb{R},\ 0 \in \mathbb{R})$

G2: ✓

G3: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} x_1+e_1 \\ 0 \end{pmatrix}$ => keine Gruppe

=> kein VR

(90) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2+y_2 \end{pmatrix}$; $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \cdot x_2 \end{pmatrix}$

$\langle \mathbb{R}^2, + \rangle$:

G3: $x_1 + e_1 = 0$ => keine Gruppe

=> kein VR

(91) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2+y_1 \\ x_1+y_2 \end{pmatrix}$; $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ \lambda \cdot x_2 \end{pmatrix}$

$\langle \mathbb{R}^2, + \rangle$:

G1: ✓

G2: ✓

G3: keine Gruppe, da $x_1 + e_1 = x_2 \neq x_1$ => kein VR

(92) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_2 \\ x_2+y_1 \end{pmatrix}$; $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ \lambda \cdot x_2 \end{pmatrix}$

$\langle \mathbb{R}^2, + \rangle$:

G1: ✓

G2: ✓

~~G3: keine Gruppe, da $x_2 + e_1 \ldots$~~

G5: nicht kommutativ

(93) $\langle \mathbb{R}^2, + \rangle$: G3: kein Einheitsel., keine Gruppe, kein VR

(94) the same

(95) $V = \langle \mathbb{R}^3, +, \mathbb{R} \rangle \qquad W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \;\middle|\; x = 2y \right\}$

$W \leq V$ [TR] ?

$a, b \in W \qquad 1,\ a - b \in W$

$\qquad\qquad\qquad 2,\ \lambda \cdot a \in W$

~~1, $\begin{pmatrix} x_1 \\ 2x_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ 2x_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2(x_1 - x_2) \\ z_1 - z_2 \end{pmatrix} \in W$~~

~~2, $\lambda \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ 2x \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ 2 \cdot \lambda x \\ \lambda z \end{pmatrix}$~~

1, $\begin{pmatrix} 2y_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} 2y_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2(y_1 - y_2) \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix} \in W$

2, $\lambda \cdot \begin{pmatrix} 2y \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\lambda y \\ \lambda y \\ \lambda z \end{pmatrix} \in W \qquad \Rightarrow \underline{W \leq V \text{ [TR]}}$

(96) $\langle \mathbb{R}^3, +, \mathbb{R} \rangle \qquad W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \;\middle|\; y = -z \right\}$

1, $\begin{pmatrix} x_1 \\ -z_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ -z_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -(z_1 + z_2) \\ z_1 + z_2 \end{pmatrix} \in W \qquad$ aufschreiben!

2, $\lambda \cdot \begin{pmatrix} x \\ -z \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ -\lambda \cdot z \\ \lambda \cdot z \end{pmatrix} \in W \qquad \Rightarrow \underline{W \leq V \text{ [TR]}}$

(97) $\langle \mathbb{R}^3, +, \mathbb{R} \rangle \qquad W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \;\middle|\; x + y + z = 0 \right\}$

1, $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad$ aufschreiben!

$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) =$

$= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0 \qquad \Rightarrow \in W$

2, $\lambda \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot z \end{pmatrix}$

$\lambda \cdot x + \lambda \cdot y + \lambda \cdot z = \lambda \cdot (x + y + z) = \lambda \cdot 0 = 0 \qquad \Rightarrow \in W$

$\Rightarrow \underline{W \leq V \text{ [TR]}}$

98) $\langle \mathbb{R}^3, \bar{+}, \mathbb{R} \rangle$ $\quad W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V / xy = 0 \right\}$

1) $\dots\dots = \begin{pmatrix} x_1 \bar{+} x_2 \\ y_1 \bar{+} y_2 \\ z_1 \bar{+} z_2 \end{pmatrix}$

$(x_1 \bar{+} x_2)(y_1 \bar{+} y_2) = \underbrace{x_1 \cdot y_1}_{0} \bar{+} x_2 \cdot y_1 \bar{+} x_1 \cdot y_2 + \underbrace{x_2 \cdot y_2}_{0} =$

$= - x_2 \cdot y_1 \bar{+} x_1 \cdot y_2 \neq 0$

$W \notin V$ [TR]

99)

(99) $\langle V, +, \mathbb{R} \rangle$ $\quad f: \mathbb{R} \to \mathbb{R}$

$\langle W, +, \mathbb{R} \rangle$ $\quad f: \mathbb{R} \to \mathbb{R}$; $f(x) = -f(-x)$

(101) $\mathbb{Q}[\sqrt{5}] = \{a + b \cdot \sqrt{5} \mid a, b \in \mathbb{R}\}$; $\langle \mathbb{Q}[\sqrt{5}], +, \mathbb{Q} \rangle$

$\langle \mathbb{Q}[\sqrt{5}], + \rangle$ abel'sche Gruppe

$\lambda \cdot v$:

$a = c + d \cdot \sqrt{5}$

$b = e + f \cdot \sqrt{5}$

1, $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$

$\lambda \cdot (c + d \cdot \sqrt{5} + e + f \cdot \sqrt{5}) = \lambda \cdot (c + d\sqrt{5}) + \lambda \cdot (e + f\sqrt{5})$

2, $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

$(\lambda + \mu) \cdot (c + d\sqrt{5}) = \lambda \cdot (c + d \cdot \sqrt{5}) + \mu \cdot (c + d \cdot \sqrt{5})$

3, $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

$\lambda \cdot (\mu \cdot (c + d\sqrt{5})) = (\lambda \cdot \mu) \cdot (c + d\sqrt{5})$

4, ✓

=> <u>VR</u>

(102) ✓

(103) ~~$\lambda \cdot r = r$ $\quad$ f.a. $\lambda \in K$ $\quad \lambda \cdot r = r$~~

~~$r = r + r$ $\quad \lambda \cdot (r + r) = r$~~

~~$r = \lambda \cdot a + (-\lambda \cdot a)$ $\quad \lambda \cdot r + \lambda \cdot r = r$~~

~~$r = a \cdot (\lambda + (-\lambda))$ $\quad \lambda \cdot r + \lambda \cdot r = \lambda \cdot r$~~

~~$r = a \cdot 0$~~

$$0 = 0$$
$$0 = 0 + 0$$
$$0 = \lambda \cdot a + (-\lambda) \cdot a$$
$$0 = (\lambda - \lambda)\, a$$
$$0 = 0 \cdot a$$

$$1 \cdot a = a$$
$$a = a + 0$$

---

108 VR (blabla)

109 kleinster TR: $L\left(\left\{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\}\right) =$

$$= \left\{\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\x_4\end{pmatrix} \middle| \; x_0, x_2, x_4 = 0\right\}$$

110 kleinster TR: $L\left(\left\{\begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}\right\}\right) =$

$$= \left\{\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\x_4\end{pmatrix} \middle| \; x_0, x_4 = 0;\; x_1 = -x_2 + x_3\right\}$$

111 VR (blabla)

112 kleinster TR: $L\left(\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}\right) =$

$$= \left\{\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix} \middle| \; x_0, x_3 = 0\right\}$$

113 kleinster TR: $L\left(\left\{\begin{pmatrix}0\\0\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\end{pmatrix}\right\}\right) =$

$$= \left\{\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix} \middle| \; x_0, x_1 = 0\right\}$$

(114) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x+y+z \leq 0 \right\} \subseteq \mathbb{R}^3$ [TR] ?

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$$(x_1 - x_2) + (y_1 - y_2) + (z_1 - z_2) \leq 0$$

$$\underbrace{(x_1 + y_1 + z_1)}_{\leq 0} - \underbrace{(x_2 + y_2 + z_2)}_{\leq 0} \leq 0 \quad \Rightarrow \underline{\text{kein TR}}$$

(115) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x+y+z \geq 0 \right\} \subseteq \mathbb{R}^3$ [TR] ?

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$$(x_1 - x_2) + (y_1 - y_2) + (z_1 - z_2) \geq 0$$

$$\underbrace{(x_1 + y_1 + z_1)}_{\geq 0} - \underbrace{(x_2 + y_2 + z_2)}_{\geq 0} \geq 0 \quad \Rightarrow \underline{\text{kein TR}}$$

(116.)

⋮

$$\underbrace{(x_1 + y_1 + z_1)}_{=0} - \underbrace{(x_2 + y_2 + z_2)}_{=0} = 0$$

$$\lambda \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot z \end{pmatrix}$$

$$\lambda \cdot x + \lambda \cdot y + \lambda \cdot z = \lambda \cdot \underbrace{(x+y+z)}_{=0} = 0 \quad \Rightarrow \underline{\text{TR}}$$

(117) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x = 2z \right) \subseteq \mathbb{R}^3$ [TR] ?

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$$x_1 - x_2 = 2(z_1 - z_2)$$

$$x_1 - x_2 = 2z_1 - 2z_2$$

=

=

$$\lambda \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot z \end{pmatrix}$$

$\lambda \cdot x = 2 \cdot \lambda z$

$x = 2 \cdot z$ $\quad \Rightarrow \quad W \subseteq \mathbb{R}^3$ [TR] ✓

(118) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x = -z \right\}$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$x_1 - x_2 = z_2 - z_1 \qquad x_2 = -z_2 \, ; \; x_1 = -z_1$

LS: $\quad -z_1 + z_2 = z_2 - z_1$ ✓

$$\lambda \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot z \end{pmatrix}$$

$\lambda \cdot x = -\lambda \cdot z \qquad x = -z$

$-\lambda \cdot x = -\lambda \cdot z \qquad \Rightarrow W \subseteq \mathbb{R}^3$ [TR]

(119) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, xy = 0 \right\}$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$(x_1 - x_2)(y_1 - y_2) = \underbrace{x_1 y_1}_{0} - x_2 y_1 - x_1 y_2 + \underbrace{x_2 y_2}_{0} =$

$= x_2 y_1 - x_1 y_2$ nicht 0 f. a. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

$\Rightarrow W \nsubseteq \mathbb{R}^3$ [TR]

(120) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x^2 + y^2 = 1 \right\}$

$$\begin{pmatrix} 0,6 \\ 0,8 \\ 0 \end{pmatrix} \begin{pmatrix} 0,8 \\ 0,6 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

$0,8 \cdot 0,6 + 0,6 \cdot 0,8 = \frac{1}{2}$ f. A.

$\boxed{x_1 x_2 + y_1 y_2 = \frac{1}{2}}$

$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 1$

$x_1^2 - 2x_1 x_2 + x_2^2 + y_1^2 - 2y_1 y_2 + y_2^2 = 1$

$2 - 2x_1 x_2 - 2y_1 y_2 = 1$

$-2x_1 x_2 - 2y_1 y_2 = -1 \qquad \Rightarrow W \nsubseteq \mathbb{R}^3$ [TR]

(121)

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 2 \\ 4 & 1 & 1 \end{pmatrix} = \cancel{4} + 16 + 8 - 64 - 2 - \cancel{4} = -42$$

$\Rightarrow$ Vektoren l.u.

$|B| = 3$; $\dim \mathbb{R}^3 = 3 \Rightarrow$ Basis

$\dim K^p = p$

(122)

$$\lambda_1 \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

I. $\lambda_1 + 2\lambda_2 + 4\lambda_3 = -1$

II. $2\lambda_1 + 4\lambda_2 + 2\lambda_3 = 0$

III. $4\lambda_1 + \lambda_2 + \lambda_3 = 1$

$$\begin{pmatrix} 1 & 2 & 4 & -1 \\ 2 & 4 & 2 & 0 \\ 4 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 & -1 \\ 0 & 0 & -6 & 2 \\ 0 & -7 & -15 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 & -1 \\ 0 & -7 & -15 & 5 \\ 0 & 0 & -6 & 2 \end{pmatrix}$$

$-6\lambda_3 = 2$

$\lambda_3 = -\frac{1}{3}$

$\cancel{-7 \cdot (-15)}$

$-7 \cdot \lambda_2 - \cancel{15}^{5} \cdot \left(-\frac{1}{\cancel{3}}\right) = 5$

$-7\lambda_2 = 0$

$\lambda_2 = 0$

$\cancel{\lambda_1 + 2 \cdot \left(-\frac{10}{7}\right) + 4 \cdot \left(-\frac{1}{3}\right) = -1}$

$\cancel{\lambda_1 - \frac{20}{7} - \frac{4}{3} = -1}$

$\cancel{\lambda_1 = -\frac{21}{21} + \frac{60}{21} + \frac{28}{21} = \frac{67}{21}}$

$\lambda_1 + 0 + 4 \cdot \left(-\frac{1}{3}\right) = -1$

$\lambda_1 = \frac{1}{3}$

$$B' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} -1 & 2 & 4 & 1 \\ 0 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 4 & 1 \\ 0 & 4 & 2 & 2 \\ 0 & 3 & 5 & 2 \end{pmatrix}$$

$$\begin{array}{cccc} 0 & 3 & 5 & 2 \\ 1 \cdot \frac{3}{4} - 0 & -3 & -\frac{3}{2} & -\frac{3}{2} \\ \hline 0 & 0 & \frac{7}{2} & \frac{1}{2} \end{array}$$

$$\begin{pmatrix} -1 & 2 & 4 & 1 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & \frac{7}{2} & \frac{1}{2} \end{pmatrix}$$

$\frac{7}{2} \cdot \lambda_3 = \frac{1}{2}$	$4 \cdot \lambda_2 + \frac{2}{7} = 2$	$-\lambda_1 + \frac{6}{7} + \frac{4}{7} = 1$
$7\lambda_3 = 1$	$4 \cdot \lambda_2 = \frac{12}{7}$	$-\lambda_1 = -\frac{3}{7}$
$\underline{\lambda_3 = \frac{1}{7}}$	$\underline{\lambda_2 = \frac{3}{7}}$	$\underline{\lambda_1 = \frac{3}{7}}$

$$\begin{pmatrix} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 0 & 3 & 5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & \frac{7}{2} & \frac{5}{2} \end{pmatrix}$$

$\frac{7}{2} \lambda_3 = \frac{5}{2}$	$4\lambda_2 + \frac{10}{7} = 2$	$-\lambda_1 + \frac{2}{7} + \frac{20}{7} = 3$
$7\lambda_3 = 5$	$4\lambda_2 = \frac{4}{7}$	$-\lambda_1 = -\frac{1}{7}$
$\underline{\lambda_3 = \frac{5}{7}}$	$\underline{\lambda_2 = \frac{1}{7}}$	$\underline{\lambda_1 = \frac{1}{7}}$

$$\Rightarrow \quad B'' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$$

122. $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$ $\quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$\lambda_1 \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 + 2 \cdot \lambda_2 + 4 \cdot \lambda_3 = -1$$
$$2\lambda_1 + 4 \cdot \lambda_2 + 2 \cdot \lambda_3 = 0$$
$$4\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 2 & 4 & 2 & 0 \\ 4 & 1 & 1 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & 0 & -6 & 2 \\ 0 & -7 & -15 & 5 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -7 & -15 & 5 \\ 0 & 0 & -6 & 2 \end{array} \right)$$

$$-6 \cdot \lambda_3 = 2$$
$$\underline{\lambda_3 = -\frac{1}{3}}$$

$$-7\lambda_2 + (\cancel{-15}^{5}) \cdot (-\frac{1}{\cancel{3}}) = \cancel{4}\ 5$$
$$-7\lambda_2 = 0$$
$$\underline{\lambda_2 = 0}$$

$$\lambda_1 - \frac{4}{3} = -1$$
$$\underline{\lambda_1 = \frac{1}{3}}$$

~~usw.~~ $\Rightarrow B' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$

$$\lambda_1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$-\lambda_1 + 2\lambda_2 + 4\lambda_3 = 3$$
$$4\lambda_2 + 2\lambda_3 = 2$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right) \qquad \left(\begin{array}{ccc|c} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 0 & 3 & 5 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 4 & 3 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & \frac{7}{2} & \frac{5}{2} \end{array} \right)$$

122. Fortsetzung: $\frac{7}{2}\lambda_3 = \frac{5}{2}$ | $4\lambda_2 + \frac{10}{7} = \frac{14}{7}$

$\underline{\lambda_3 = \frac{5}{7}}$ | $\underline{\lambda_2 = \frac{1}{7}}$

$-\lambda_1 + \frac{2}{7} + \frac{20}{7} = \frac{21}{7}$ $\Rightarrow$ $\underline{\lambda_1 = +\frac{1}{7}}$

$\Rightarrow$ $\underline{B^4 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right.}$

123. $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \right\}$

1) $\lambda_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$$
$$\lambda_1 + 3\lambda_2 + 3\lambda_3 = 0$$
$$3\lambda_2 + 2\lambda_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 3 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$\lambda_3 = 0$, $\lambda_2 = 0$, $\lambda_1 = 0$

Vektoren l.u.

2) $|B| = \dim \mathbb{R}^3 = 3$ ✓

124. $B = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\3\end{pmatrix}, \begin{pmatrix}3\\3\\2\end{pmatrix} \right\}$; $\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 1 & 3 & 3 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

$\lambda_3 = -1 \qquad \lambda_2 = 1$

$\lambda_1 + 2 - 3 = -1 \qquad \lambda_1 = 0$

$\Rightarrow B' = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\3\\2\end{pmatrix} \right\}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$\lambda_3 = 0$; $\lambda_2 = 1$;

$\lambda_1 - 1 = 1$; $\lambda_1 = 2$

$\Rightarrow \underline{B'' = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\3\\2\end{pmatrix} \right\}}$

125.)

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -5 & -8 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 12 & 0 \end{array} \right)$$

$$\lambda_3 = 0 \; ; \; \lambda_2 = 0 \; ; \; \lambda_1 = 0$$

$\Rightarrow$ Vektoren l.u.

$|B| = \dim \mathbb{R}^3 = 3$

$\Rightarrow$ Basis

126.)

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\} ; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 1 & 1 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & -1 & -4 & 2 \\ 0 & -5 & -8 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & 12 & -6 \end{array} \right)$$

$$\lambda_3 = -\frac{1}{2} \; ;$$

$$-\lambda_2 + 2 = 2$$

$$\lambda_2 = 0$$

$$\lambda_1 \cancel{A} - \frac{3}{2} = -1 \quad \rightarrow \quad \lambda_1 = \frac{1}{2}$$

126. Fortsetzung:

$$B' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 2 \\ 0 & 3 & \cancel{8}\,4 & 5 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & \cancel{1}\,2 & 3 \end{array} \right)$$

$\cancel{\lambda_3 = 1}$

$\cancel{3\lambda_2 + 2 = 1}$

$\cancel{\lambda_2 = -\frac{1}{3}}$

$$\lambda_3 = \frac{3}{2}$$

$$3\lambda_2 + 3 = 2$$

$$3\lambda_2 = -1$$

$$\lambda_2 = -\frac{1}{3}$$

$\cancel{-\lambda_1 + \frac{2}{3} + 6 = 4}$

$\cancel{-\lambda_1 = \frac{1}{3}}$

$\cancel{-\lambda_1 - \frac{2}{3} + \frac{9}{2} = 4}$

$\cancel{\lambda_1 = \frac{9}{2} - \frac{8}{2} + \frac{4}{6} = \frac{27}{6} - \frac{24}{6} + \frac{4}{6} = \frac{7}{6}}$

I $\lambda_1 - \frac{1}{3} + \frac{3}{2} = 1$

$$\lambda_1 = -\frac{1}{6}$$

$$\Rightarrow \quad B' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

127.) e_1, e_2, e_3 l.n. $\Leftrightarrow$ $e_1 + e_2,\ e_2 + e_3,\ e_3$ l.n.

$\lambda_1 \cdot e_1 + \lambda_2 \cdot e_2 + \lambda_3 \cdot e_3 = o \quad \Leftrightarrow \quad \lambda_1, \lambda_2, \lambda_3 = 0$

$\mu_1 \cdot (e_1 + e_2) + \mu_2 \cdot (e_2 + e_3) + \mu_3 \cdot e_3 = o$

$\mu_1 \cdot e_1 + (\mu_1 + \mu_2) \cdot e_2 + (\mu_2 + \mu_3) e_3 = o$

$\lambda_1 = 0 \Rightarrow \mu_1 = 0$

$\lambda_2 = 0 \Rightarrow \mu_1 + \mu_2 = 0$

$0 + \mu_2 = 0 \Rightarrow \mu_2 = 0$

$\lambda_3 = 0 \Rightarrow \mu_2 + \mu_3 = 0$

$0 + \mu_3 = 0 \Rightarrow \mu_3 = 0$

$\mu_1 = 0 \Rightarrow \lambda_1 = 0$

$\mu_2 = 0 \Rightarrow \mu_1 + \mu_2 = 0 \Rightarrow \lambda_2 = 0$

$\mu_3 = 0 \Rightarrow \mu_2 + \mu_3 = 0 \Rightarrow \lambda_3 = 0$

128.) e_1, e_2, e_3 l.n. $\Leftrightarrow$ $e_1 + e_2 + e_3,\ e_2 + e_3,\ e_3$ l.n.

$\lambda_1 \cdot e_1 + \lambda_2 \cdot e_2 + \lambda_3 \cdot e_3 = 0$

$\mu_1 \cdot (e_1 + e_2 + e_3) + \mu_2 (e_2 + e_3) + \mu_3 e_3 = 0$

$\mu_1 \cdot e_1 + (\mu_1 + \mu_2) e_2 + (\mu_1 + \mu_2 + \mu_3) e_3 = 0$

$\lambda_1 = 0 \Rightarrow \mu_1 = 0$

$\lambda_2 = 0 \Rightarrow \mu_1 + \mu_2 = 0 \Rightarrow \mu_2 = 0$

$\lambda_3 = 0 \Rightarrow \mu_1 + \mu_2 + \mu_3 = 0 \Rightarrow \mu_3 = 0$

$\mu_1 = 0 \Rightarrow \lambda_1 = 0$

$\mu_2 = 0 \Rightarrow \mu_1 + \mu_2 = 0 \Rightarrow \lambda_2 = 0$

$\mu_3 = 0 \Rightarrow \mu_1 + \mu_2 + \mu_3 = 0 \Rightarrow \lambda_3 = 0$

129.) v_1, v_2, v_3 l.u. $\Leftrightarrow$ $v_1 - v_2, v_2, v_2 - v_3$ l.u.

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3 = 0$$

$$\mu_1 (v_1 - v_2) + \mu_2 \cdot v_2 + \mu_3 (v_2 - v_3) = 0$$

$$\mu_1 \cdot v_1 + (\mu_2 - \mu_1 + \mu_3) \cdot v_2 - \mu_3 \cdot v_3 = 0$$

$\lambda_1 = 0 \Rightarrow \underline{\mu_1 = 0}$

$\lambda_3 = 0 \Rightarrow \mu_3 = 0$

$\lambda_2 = 0 \Rightarrow \mu_2 - \mu_1 + \mu_3 = 0 \Rightarrow \underline{\mu_2 = 0}$

$\mu_1 = 0 \Rightarrow \underline{\lambda_1 = 0}$

$\mu_3 = 0 \Rightarrow \underline{\lambda_3 = 0}$

$\mu_2 = 0 \Rightarrow \mu_2 - \mu_1 + \mu_3 = 0 \Rightarrow \underline{\lambda_2 = 0}$

130.)

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7x_1 + 5x_2 \\ x_1 - 2x_3 \end{pmatrix}$$

$$A(\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2) = \lambda_1 \cdot A(v_1) + \lambda_2 \cdot A(v_2)$$

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad v_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

LS:

$$A(\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2) = A\left(\begin{pmatrix} \lambda_1 x_1 \\ \lambda_1 x_2 \\ \lambda_1 x_3 \end{pmatrix} + \begin{pmatrix} \lambda_2 y_1 \\ \lambda_2 y_2 \\ \lambda_2 y_3 \end{pmatrix} \right) =$$

$$= A\begin{pmatrix} \lambda_1 x_1 + \lambda_2 y_1 \\ \lambda_1 x_2 + \lambda_2 y_2 \\ \lambda_1 x_3 + \lambda_2 y_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 7(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) \\ (\lambda_1 x_1 + \lambda_2 y_1) - 2(\lambda_1 x_3 + \lambda_2 y_3) \end{pmatrix}$$

130.) Fortsetzung:

RS:

$\lambda_1 \cdot A(x_1) + \lambda_2 \cdot A(x_2) =$

$= \lambda_1 \cdot A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \lambda_2 \cdot A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =$

$= \lambda_1 \cdot \begin{pmatrix} 7x_1 + 5x_2 \\ x_1 - 2x_3 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 7y_1 + 5y_2 \\ y_1 - 2y_3 \end{pmatrix} =$

$= \begin{pmatrix} \lambda_1 \cdot 7x_1 + \lambda_1 \cdot 5x_2 + \lambda_2 \cdot 7y_1 + \lambda_2 \cdot 5y_2 \\ \lambda_1 \cdot x_1 - \lambda_1 \cdot 2x_3 + \lambda_2 \cdot y_1 - \lambda_2 \cdot 2y_3 \end{pmatrix} =$

$= \begin{pmatrix} 7(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) \\ (\lambda_1 x_1 + \lambda_2 y_1) - 2(\lambda_1 x_3 + \lambda_2 y_3) \end{pmatrix}$

LS = RS $\Rightarrow$ lineare Abbildung

131.) $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 5x_2 \\ x_1 - 3x_3 \end{pmatrix}$

LS: $A\begin{pmatrix} \lambda_1 x_1 + \lambda_2 y_1 \\ \lambda_1 x_2 + \lambda_2 y_2 \\ \lambda_1 x_3 + \lambda_2 y_3 \end{pmatrix} = \begin{pmatrix} 3(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) \\ (\lambda_1 x_1 + \lambda_2 y_1) - 3(\lambda_1 x_3 + \lambda_2 y_3) \end{pmatrix}$

RS: $\lambda_1 \cdot A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \lambda_2 \cdot A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =$

$= \lambda_1 \cdot \begin{pmatrix} 3x_1 + 5x_2 \\ x_1 - 3x_3 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 3y_1 + 5y_2 \\ y_1 - 3y_3 \end{pmatrix} =$

$= \begin{pmatrix} 3(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) \\ (\lambda_1 x_1 + \lambda_2 y_1) - 3(\lambda_1 x_3 + \lambda_2 y_3) \end{pmatrix}$

LS = RS $\Rightarrow$ lineare Abbildung

132. $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 5x_2 - x_3 \\ -3x_2 \end{pmatrix}$

LS: $A\begin{pmatrix} \lambda_1 x_1 + \lambda_2 y_1 \\ \lambda_1 x_2 + \lambda_2 y_2 \\ \lambda_1 x_3 + \lambda_2 y_3 \end{pmatrix} = \begin{pmatrix} 3(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) - (\lambda_1 x_3 + \lambda_2 y_3) \\ -3(\lambda_1 x_2 + \lambda_2 y_2) \end{pmatrix}$

RS: $\lambda_1 \cdot A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \lambda_2 \cdot A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =$

$= \lambda_1 \cdot \begin{pmatrix} 3x_1 + 5x_2 - x_3 \\ -3x_2 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 3y_1 + 5y_2 - y_3 \\ -3y_2 \end{pmatrix} =$

$= \begin{pmatrix} 3(\lambda_1 x_1 + \lambda_2 y_1) + 5(\lambda_1 x_2 + \lambda_2 y_2) - (\lambda_1 x_3 + \lambda_2 y_3) \\ -3(\lambda_1 x_2 + \lambda_2 y_2) \end{pmatrix}$

LS = RS $\Rightarrow$ lineare Abbildung

133. $V = \mathbb{C}^3$

$U = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V \middle| z_1 + z_2 = z_3 \right\} = L\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}\right)$

$W = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V \middle| z_2 = -z_1 \right\} = L\left(\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right)$

$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (z_1 + z_2) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

$= z_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + z_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z_3 \end{pmatrix}$

dim $U = 2$

dim $W = 2$

133. Fortsetzung:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - \begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} z_1 - z_1' \\ z_2 - z_2' \\ z_3 - z_3' \end{pmatrix}$$

$z_1 - z_1' + z_2 - z_2' = z_3 - z_3'$

$z_3 - z_3' = z_3 - z_3'$ ✓

$$\lambda \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda z_1 \\ \lambda z_2 \\ \lambda z_3 \end{pmatrix}$$

$\lambda z_1 + \lambda z_2 = \lambda z_3$

$\lambda (z_1 + z_2) = \lambda z_3$

$\lambda z_3 = \lambda z_3$ ✓ $\Rightarrow U \subseteq V$ [TR]

$U \subseteq V$ [TR]

1, $a, b \in U$

$a - b \in U$

2, $a \in U, \lambda \in K$

$\lambda \cdot a \in U$

$\Rightarrow U \subseteq V$ [TR]

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - \begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} z_1 - z_1' \\ z_2 - z_2' \\ z_3 - z_3' \end{pmatrix}$$

~~$z_1 \, z_1$~~

$z_2 - z_2' = z_1' - z_1$

$-z_1 + z_1' = z_1' - z_1$ ✓

$$\lambda \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda \cdot z_1 \\ \lambda \cdot z_2 \\ \lambda \cdot z_3 \end{pmatrix}$$

$\lambda \cdot z_2 = -\lambda \cdot z_1$

$-\lambda \cdot z_1 = -\lambda \cdot z_1$ ✓ $\Rightarrow W \subseteq V$[TR]

134. $U + W = \{ \cancel{\ldots}\ u + w \mid u \in U,\ w \in W \}$

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\in V} = \underbrace{\begin{pmatrix} x \\ y \\ x+y \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ z-(x+y) \end{pmatrix}}_{\in W}$$

$\Rightarrow V = U + W \quad \Rightarrow \dim V = \dim (U+W)$

$\Rightarrow \underline{\dim (U+W) = 3}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ x_1+y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ \cancel{z_1}\ z_2 \end{pmatrix} =$$

$x = x_1 + x_2$

$y = y_1 + y_2$

$z = x_1 + y_1 + z_2$

$z_2 = z - (x_1 + y_1)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ z-(x+y) \end{pmatrix}}_{\in W = V}$$

$$U \cap W = \left\{ \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} \in V \right\} = L\left(\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \right)$$

$\Rightarrow \dim (U \cap W) = 1$

$\dim (U+W) = \dim U + \dim W - \dim (U \cap W)$

$3 = 2 + 2 - 1$ ✓

$$U \cap W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \mid y = -x \text{ und } z = 0 \right\}$$

$$\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - x \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= x \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 = x \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

135. $U = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V \middle| z_1 - z_2 = z_3 \right\}$ $\quad V = \mathbb{C}^3$

$W = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V \middle| z_2 = z_1 \right\}$

$U \subseteq V$ [TR] ?

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - \begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} z_1 - z_1' \\ z_2 - z_2' \\ z_3 - z_3' \end{pmatrix}$$

$$z_1 - z_1' - z_2 + z_2' = z_3 - z_3'$$

$z_3 = z_1 - z_2$; $z_3' = z_1' - z_2'$

$$z_3 - z_3' = z_3 - z_3' \checkmark$$

$$\lambda \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda \cdot z_1 \\ \lambda \cdot z_2 \\ \lambda \cdot z_3 \end{pmatrix}$$

$\lambda \cdot z_1 - \lambda \cdot z_2 = \lambda \cdot z_3$

$\lambda \cdot (z_1 - z_2) = \lambda \cdot z_3$

$z_1 - z_2 = z_3 \Rightarrow \lambda \cdot z_3 = \lambda \cdot z_3 \checkmark \Rightarrow \underline{U \subseteq V \text{ [TR]}}$

$W \subseteq V$ [TR] ?

⋮

$z_2 - z_2' = z_1 - z_1'$

$z_2 = z_1$; $z_2' = z_1' \Rightarrow z_1 - z_1' = z_1 - z_1' \checkmark$

⋮

$\lambda \cdot z_2 = \lambda \cdot z_1$

$z_2 = z_1 \Rightarrow \lambda \cdot z_1 = \lambda \cdot z_1 \checkmark \Rightarrow \underline{W \subseteq V \text{ [TR]}}$

$U = L\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}\right)$ $\quad \underline{\dim U = 2}$

$W = L\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}\right)$ $\quad \underline{\dim W = 2}$

136. $\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\in V} = \underbrace{\begin{pmatrix} x \\ y \\ x-y \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ z-(x-y) \end{pmatrix}}_{\in W}$

$V = U + W \qquad \dim V = 3 \Rightarrow \dim (U+W) = \dim V = 3$

$\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in U \cap W ; \; x \in \mathbb{C}$

$\Rightarrow W = L\left(\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}\right) \Rightarrow \dim W = 1$

$\dim (U+W) = \dim U + \dim W - \dim (U \cap W)$

$3 = 2 + 2 - 1$ ✓

137. $V = \mathbb{C}^3$

$U = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V / z_1 = 2z_2 = 3z_3 \right\}$

$W = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V / z_2 = 0 \right\}$

$U \subseteq V$ [TR]?

⋮

1, $z_1 - z_1' = 2(z_2 - z_2') = 3(z_3 - z_3')$

$z_1 = 2z_2 ; \; z_1' = 2z_2'$

$\Rightarrow 2z_2 - 2z_2' = 2z_2 - 2z_2'$ ✓

$2z_2 = 3z_3 ; \; 2z_2' = 3z_3'$

$\Rightarrow 3z_3 - 3z_3' = 3z_3 - 3z_3'$ ✓

2,

⋮

$\lambda \cdot z_1 = 2\lambda z_2 = 3\lambda z_3$

$z_1 = 2z_2 \Rightarrow 2\lambda z_2 = 2\lambda z_2$ ✓

$2z_2 = 3z_3 \qquad 3\lambda z_3 = 3\lambda z_3$ ✓ $\Rightarrow U \subseteq V$ [TR]

137. Fortsetzung

$W \subseteq V[\mathbb{R}]$?

1, $\quad z_2 - z_2' = 0$

$\quad z_2 = 0; \; z_2' = 0 \quad \Rightarrow \quad 0 - 0 = 0$ ✓

2, $\quad \lambda \cdot z_2 = 0$

$\quad z_2 = 0 \quad \Rightarrow \quad 0 = 0$ ✓

$\Rightarrow \quad W \subseteq V[\mathbb{R}]$

$U = L\left(\left\{\begin{pmatrix}6\\3\\2\end{pmatrix}\right\}\right) \qquad \dim U = 1$

$W = L\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right) \qquad \dim W = 2$

138. $\quad U \cap W = \cancel{\emptyset}\{0\} \Rightarrow \dim(U \cap W) = \cancel{1}\, 0$

$U \cap W = \left\{\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix} \in V \,/\, z_1 = 2z_2 = 3z_3 \text{ und } z_2 = 0\right\}$

$z_2 = 0 \Rightarrow z_1 = 0 \Rightarrow z_3 = 0$

$\Rightarrow \quad U \cap W = \{0\}$

$U + W = \{\underbrace{(u+w)}_{} \overset{\in V}{/}\; u \in U;\; w \in W\}$

$u + w = \underbrace{\lambda_1 \cdot \begin{pmatrix}6\\3\\2\end{pmatrix}}_{\in U} + \underbrace{\lambda_2 \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_3 \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\in W}$

$\begin{pmatrix}6\\3\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ l. u.

$\Rightarrow \quad U + W = L\left(\left\{\begin{pmatrix}6\\3\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right)$

$\Rightarrow \quad \dim(U+W) = 3$

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$3 = 1 + 2 - 0$ ✓

139. lineare Abbildung: $A\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

$$A\left(\lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix}\right) = \underbrace{\lambda\cdot A\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot A\begin{pmatrix}2\\3\end{pmatrix}}_{\lambda\cdot\begin{pmatrix}1\\-2\end{pmatrix} + \mu\cdot\begin{pmatrix}1\\-2\end{pmatrix}}$$

$$\lambda\cdot\begin{pmatrix}1\\-2\end{pmatrix} + \mu\cdot\begin{pmatrix}1\\-2\end{pmatrix} = 0 \qquad (\lambda+\mu)\begin{pmatrix}1\\-2\end{pmatrix} = 0$$

$$\lambda+\mu = 0 \quad \to \quad \lambda = -\mu$$

$$\lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix} = \lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} - \lambda\cdot\begin{pmatrix}2\\3\end{pmatrix} = \lambda\begin{pmatrix}-1\\-3\end{pmatrix}$$

$$\mathrm{Ker}\ A = L\left(\left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}\right) \qquad \lambda\cdot\left[\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}\right]$$

$$\dim(\mathrm{Ker}\ A) = 1$$

$$A\Big(\underbrace{\lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix}}_{\lambda+\mu=0}\Big) = 0$$

---

140. $a \in \mathbb{R}^2$

$$a = \lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix}$$

$$A(a) = (\lambda+\mu)\begin{pmatrix}1\\-2\end{pmatrix} \qquad A(\mathbb{R}^2) = L\left(\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}\right)$$

$\Rightarrow$ ~~$A(\mathbb{R}^2)$~~ ~~Basis von $A(\mathbb{R}^2) = \left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$~~

$$\text{Rang}\ A = \dim A(\mathbb{R}^2) = 1$$

$$\dim(\mathrm{Ker}\ A) + \text{Rang}\ A = \dim \mathbb{R}^2$$

$$1 + 1 = 2 \ \checkmark$$

141. $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

$$A\begin{pmatrix}0\\1\end{pmatrix} = \cancel{A\left(\lambda\cdot\begin{pmatrix}1\\-2\end{pmatrix} + \mu\cdot\begin{pmatrix}1\\-2\end{pmatrix}\right)}$$

$$= A\left(\lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix}\right)$$

$$\lambda\cdot\begin{pmatrix}1\\0\end{pmatrix} + \mu\cdot\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\lambda + 2\mu = 0$$

$$3\mu = 1 \quad \Rightarrow \quad \mu = \frac{1}{3}$$

$$\lambda + \frac{2}{3} = 0 \quad \Rightarrow \quad \lambda = -\frac{2}{3}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = -\frac{2}{3}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{3}\cdot\begin{pmatrix}2\\3\end{pmatrix}$$

$$A\begin{pmatrix}0\\1\end{pmatrix} = -\frac{2}{3}\cdot A\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{3}\cdot A\begin{pmatrix}2\\3\end{pmatrix} =$$

$$= -\frac{2}{3}\cdot\begin{pmatrix}1\\-2\end{pmatrix} + \frac{1}{3}\cdot\begin{pmatrix}1\\-2\end{pmatrix} =$$

$$= -\frac{1}{3}\cdot\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$$

$$\Rightarrow \quad \underline{\mathfrak{A} = \begin{pmatrix}1 & -\frac{1}{3}\\ -2 & \frac{2}{3}\end{pmatrix}}$$

142. $A\begin{pmatrix}0\\1\end{pmatrix} = A\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\lambda \cdot \begin{pmatrix}1\\-1\end{pmatrix} + \mu \cdot \begin{pmatrix}1\\-1\end{pmatrix} = 0$

$\lambda = -\mu$

$-\cancel{\lambda}\mu \begin{pmatrix}0\\1\end{pmatrix} + \mu \cdot \begin{pmatrix}3\\2\end{pmatrix} =$

$= \mu \left(\begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \right) = \mu \begin{pmatrix}3\\1\end{pmatrix}$

$\mathrm{Ker}\ A = L\left(\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}\right)$

$\underline{\dim \mathrm{Ker}\ A = 1}$

143. $A(\mathbb{R}^2) = L\left(\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}\right)$

$\mathrm{Rang}\ A = \dim(A(\mathbb{R}^2)) =$

$= \dim\left(L\left(\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}\right)\right) = \cancel{2}\ 1$

$\dim(\mathrm{Ker}\ A) + \mathrm{Rang}\ A = \dim \mathbb{R}^2$

$1 + 1 = 2$ ✓

144. $A\begin{pmatrix}1\\0\end{pmatrix} = ?$

$\begin{pmatrix}1\\0\end{pmatrix} = \lambda \cdot \begin{pmatrix}0\\1\end{pmatrix} + \mu \cdot \begin{pmatrix}3\\2\end{pmatrix}$

$3\mu = 1 \Rightarrow \underline{\mu = \frac{1}{3}}$

$\lambda + 2\mu = 0 \quad \lambda + \frac{2}{3} = 0 \quad \underline{\lambda = -\frac{2}{3}}$

$A\begin{pmatrix}1\\0\end{pmatrix} = A\left(-\frac{2}{3}\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}3\\2\end{pmatrix}\right) =$

$= -\frac{1}{3}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$

$\Rightarrow \quad \mathfrak{A} = \begin{pmatrix}-\frac{1}{3} & 1\\ \frac{1}{3} & -1\end{pmatrix}$

145. $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$; $A\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$\lambda \cdot \begin{pmatrix}1\\0\end{pmatrix} + \mu \cdot \begin{pmatrix}0\\1\end{pmatrix} = \sigma \quad \rightarrow \lambda = 0;\ \mu = 0$

$\Rightarrow \operatorname{Ker} A = \{\sigma\}$; $\dim(\operatorname{Ker} A) = 0$

146. $A(\mathbb{R}^2) = L\left(\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}\right)$

$\operatorname{Rang} A = \dim A(\mathbb{R}^2) = 2$

$\dim(\operatorname{Ker} A) + \operatorname{Rang} A = \dim \mathbb{R}^2$

$0 + 2 = 2$ ✓

147. $\begin{pmatrix}1\\0\end{pmatrix} = \lambda \cdot \begin{pmatrix}1\\1\end{pmatrix} + \mu \cdot \begin{pmatrix}2\\1\end{pmatrix}$

$\mu = 1;\ \lambda = -1$

$A\begin{pmatrix}1\\0\end{pmatrix} = A\left((-1) \cdot \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}\right) =$

$= (-1) \cdot \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} = \lambda \cdot \begin{pmatrix}1\\1\end{pmatrix} + \mu \cdot \begin{pmatrix}2\\1\end{pmatrix}$

$\lambda + 2\mu = 0$

$\lambda + \mu = 1$

$\mu = -1$; $\underline{\lambda = 2}$

$A\begin{pmatrix}0\\1\end{pmatrix} = A\left(2 \cdot \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix}\right) = 2 \cdot \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

$\Rightarrow \underline{\mathfrak{A} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}}$

148.

$$D\left(\sum_{k=0}^{n} a_k \cdot x^k\right) = \sum_{k=1}^{n} k \cdot a_k \cdot x^{k-1}$$

$$\lambda \cdot D \cdot \left(\sum_{k=0}^{n} a_k x^k\right) + \mu \cdot D\left(\sum_{k=0}^{n} b_k x^k\right) = \quad \leftarrow LS$$

$$= D\left(\lambda \cdot \sum_{k=0}^{n} a_k x^k + \mu \cdot \sum_{k=0}^{n} b_k x^k\right) \quad \leftarrow RS$$

LS:

$$\lambda \cdot \sum_{k=1}^{n} k \cdot a_k x^{k-1} + \mu \sum_{k=1}^{n} k b_k x^{k-1} =$$

$$= \cancel{(\lambda+\mu)} \sum_{k=1}^{n} \left(\lambda k a_k x^{k-1} + \mu k b_k x^{k-1}\right)$$

RS:

$$D\left(\cancel{(\lambda+\mu)} \sum_{k=0}^{n} (\lambda a_k x^k + \mu b_k x^k)\right) =$$

$$= D\left(\cancel{(\lambda+\mu)} \sum_{k=0}^{n} (\lambda a_k + \mu b_k) \cdot x^k\right) =$$

$$= D\left(\sum_{k=0}^{n} \cancel{(\lambda+\mu)} (\lambda a_k + \mu b_k) \cdot x^k\right) =$$

$$= \sum_{k=1}^{n} k \cdot \cancel{(\lambda+\mu)} (\lambda a_k + \mu b_k) \cdot x^{k-1} =$$

$$= \cancel{\sum_{k=1}^{n}} \cancel{(\lambda+\mu)} \sum_{k=1}^{n} k (\lambda a_k + \mu b_k) x^{k-1} =$$

$$= \cancel{(\lambda+\mu)} \sum_{k=1}^{n} \left(\lambda k a_k x^{k-1} + \mu k b_k x^{k-1}\right)$$

LS = RS $\rightarrow$ lineare Abbildung

148. Fortsetzung:

kein Monomorphismus (nicht injektiv):

z.B. $D(2) = D(3) = 0$

kein Epimorphismus (nicht surjektiv):

kein Urbild für z.B. x^n als El. des Bildraums

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149. $$E\Big(\underbrace{\sum_{k=0}^{n} a_k x^k}_{p(x)}\Big) = \underbrace{\sum_{k=0}^{n} a_k (x+1)^k}_{p(x+1)}$$

$$E(\lambda \cdot p_1(x) + \mu \cdot p_2(x)) = \lambda \cdot E(p_1(x)) + \mu \cdot E(p_2(x))$$

LS: $$E\Big(\lambda \cdot \sum_{k=0}^{n} a_k x^k + \mu \cdot \sum_{k=0}^{n} b_k x^k\Big) =$$

$$= E\Big(\sum_{k=0}^{n} \lambda a_k x^k + \sum_{k=0}^{n} \mu b_k x^k\Big) =$$

$$= E\Big(\sum_{k=0}^{n} (\lambda a_k + \mu b_k) x^k\Big) =$$

$$= \sum_{k=0}^{n} (\lambda a_k + \mu b_k)(x+1)^k$$

RS: $$\lambda \cdot E\Big(\sum_{k=0}^{n} a_k x^k\Big) + \mu \cdot E\Big(\sum_{k=0}^{n} b_k x^k\Big) =$$

$$= \lambda \cdot \sum_{k=0}^{n} a_k (x+1)^k + \mu \cdot \sum_{k=0}^{n} b_k (x+1)^k =$$

$$= \sum_{k=0}^{n} (\lambda a_k + \mu b_k)(x+1)^k$$

LS = RS -> lineare Abbildung

Monomorphismus, Epimorphismus (= Isomorphismus

150. $\mathcal{B} = \{x^0, x^1, \dots, x^n\}$

$$D(x^0) = 0$$
$$D(x^1) = x^0$$
$$D(x^2) = 2x^1$$
$$\vdots$$
$$D(x^{n-1}) = (n-1)x^{n-2}$$
$$D(x^n) = n \cdot x^{n-1}$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & n-1 & 0 \\ 0 & 0 & 0 & & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

151.

$$S\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k \frac{x^{k+1}}{k+1}$$

$$\lambda \cdot S\left(\sum_{k=0}^{n} a_k x^k\right) + \mu \cdot S\left(\sum_{k=0}^{n} b_k x^k\right) = S\left(\lambda \cdot \sum_{k=0}^{n} a_k x^k + \mu \cdot \sum_{k=0}^{n} b_k x^k\right)$$

LS: $\lambda \cdot \sum_{k=0}^{n} a_k \frac{x^{k+1}}{k+1} + \mu \cdot \sum_{k=0}^{n} b_k \frac{x^{k+1}}{k+1} =$

$$= \sum_{k=0}^{n} (\lambda a_k + \mu b_k) \frac{x^{k+1}}{k+1}$$

RS: $S\left(\cancel{\lambda} \sum_{k=0}^{n} (\lambda a_k + \mu b_k) x^k\right) =$

$$= \sum_{k=0}^{n} (\lambda a_k + \mu b_k) \frac{x^{k+1}}{k+1}$$

LS = RS $\Rightarrow$ lineare Abbildung

Monomorphismus, Epimorphismus (= Isomorphismus)

152. $S\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k (x-1)^k$

$$\lambda \cdot S\left(\sum_{k=0}^{n} a_k x^k\right) + \mu \cdot S\left(\sum_{k=0}^{n} b_k x^k\right) = S\left(\lambda \cdot \sum_{k=0}^{n} a_k x^k + \mu \cdot \sum_{k=0}^{n} b_k x^k\right)$$

LS: $\lambda \cdot \sum_{k=0}^{n} a_k (x-1)^k + \mu \cdot \sum_{k=0}^{n} b_k (x-1)^k =$

$$= \sum_{k=0}^{n} (\lambda a_k + \mu b_k)(x-1)^k$$

RS: $S\left(\sum_{k=0}^{n} (\lambda a_k + \mu b_k) x^k\right) = \sum_{k=0}^{n} (\lambda a_k + \mu b_k)(x-1)^k$

LS = RS => lineare Abbildung

Monomorphismus, Epimorphismus (= Isomorphismus)

153. $A\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=2}^{n} k(k-1) a_k x^{k-2}$

$$\lambda \cdot A\left(\sum_{k=0}^{n} a_k x^k\right) + \mu \cdot A\left(\sum_{k=0}^{n} b_k x^k\right) = A\left(\lambda \cdot \sum_{k=0}^{n} a_k x^k + \mu \cdot \sum_{k=0}^{n} b_k x^k\right)$$

LS: $\lambda \cdot \sum_{k=2}^{n} k(k-1) a_k x^{k-2} + \mu \cdot \sum_{k=2}^{n} k(k-1) b_k x^{k-2} =$

$$= \sum_{k=2}^{n} (\lambda a_k + \mu b_k) k(k-1) x^{k-2}$$

RS: $A\left(\sum_{k=0}^{n} (\lambda a_k + \mu b_k) x^k\right) = \sum_{k=2}^{n} k(k-1)(\lambda a_k + \mu b_k) x^{k-2}$

LS = RS => lineare Abbildung

153. Fortsetzung:

$$A(x^0) = 0 \quad ; \quad A(x^1) = 0$$

$$A(x^2) = 2 \quad ; \quad A(x^3) = 6x$$

$$\vdots$$

$$A(x^{n-1}) = (n-1)(n-2)x^{n-3}$$

$$A(x^n) = n(n-1)x^{n-2}$$

$$\mathcal{A} = \begin{pmatrix}
0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & \vdots & & & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & & (n-1)(n-2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & n(n-1) & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

154.

$$\left(\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & -1 & 1 \\ 5 & 1 & -3 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 2 \\ 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & +\frac{1}{3} & \frac{4}{3} & -\frac{7}{3} \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -2 \end{array}\right) \Rightarrow \text{nicht lösbar in } \mathbb{R}$$

155. $K = \mathbb{Z}_2$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

inhomogenes System: $x_1' = 0$; $x_2' = 1$

wegen $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$

$$1x_1' + 0x_2' + 1x_3' + 1x_4' = 0$$
$$0 \cdot x_1' + 1x_2' + 0x_3' + 0x_4' = 0$$

$x_3' = 1$; $x_4' = 0$: $\quad 1x_1' + 0x_2' + 1 + 0 = 0 \Rightarrow \underline{x_1' = 1}$

$1x_2' = 0 \Rightarrow x_2' = 0$

$x_3' = 0$; $x_4 = 1$: $\quad 1x_1' + 0x_2' + 0 + 1 = 0 \Rightarrow x_1' = 1$

$1x_2' = 0 \Rightarrow x_2' = 0$

$$\underline{x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}; \quad \lambda, \mu \in \mathbb{Z}_2$$

156.

$$\left(\begin{array}{cccc|c} -3 & 1 & 2 & 1 & 2 \\ -1 & 1 & 1 & -1 & 1 \\ -5 & 1 & 3 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} -3 & 1 & 2 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{7}{3} \end{array}\right) \Rightarrow \text{nicht lösbar in } \mathbb{R}$$

157.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \Rightarrow \text{siehe 155.}$$

158.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 7 & 0 & 1 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & -\frac{5}{2} & \frac{7}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -6 & 0 \end{array}\right)$$

$x_3 = 0$

$x_2 = -1$

$x_1 = 1$

$$L = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

159.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$2x_1 + x_2 = 1 \Rightarrow 2x_1 = 2 \,;\ \underline{x_1 = 1}$

$\underline{x_2 = 2}$

159. Fortsetzung:

$x_3 = 1$:

$$\cancel{x = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda \cdot \begin{pmatrix}0\\1\\1\end{pmatrix}}$$

$$2x_1 + x_2 + 1 = 0 \qquad \underline{x_1 = 2}$$

$$x_2 + 2 = 10 \qquad \underline{x_2 = -1}$$

$$\underline{x = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda \cdot \begin{pmatrix}2\\1\\1\end{pmatrix}}$$

160.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -2 & -1 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -3 & 0 \end{array}\right) \qquad \begin{array}{l} x_3 = 0 \\ x_2 = -2 \\ x_1 = 1 \end{array} \qquad L = \left\{ \begin{pmatrix}1\\-2\\0\end{pmatrix} \right\}$$

161.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right) \qquad 1 \quad 2 \quad 2 \quad 0$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$2x_1 + x_2 = 0$$

$$\underline{x_2 = 1} \Rightarrow \underline{x_1 = 1}$$

$$2x_1 + x_2 + 1 = 0$$

$$x_2 + 2 = 10 \Rightarrow \underline{x_2 = 1}$$

$$\Rightarrow \underline{x_1 = 2}$$

$$\underline{x = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \cdot \begin{pmatrix}2\\1\\1\end{pmatrix}}$$

162.

$$\left(\begin{array}{ccc|c} 2 & 5 & -2 & 5 \\ 3 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -2 & 5 \\ 0 & -\frac{15}{2} & 4 & -\frac{7}{2} \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -2 & 5 \\ 0 & -\frac{15}{2} & 4 & -\frac{7}{2} \\ 0 & 0 & \frac{22}{15} & \frac{22}{15} \end{array}\right)$$

$\underline{x_3 = 1}$

$-\frac{15}{2}x_2 + 4 = -\frac{7}{2}$

$-\frac{15}{2}x_2 = -\frac{15}{2}$ $\quad \underline{x_2 = 1}$

$2x_1 + 5 - 2 = 5$

$2x_1 = 2$

$\underline{x_1 = 1}$

$L = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

163.

$$\left(\begin{array}{ccc|c} 2 & 5 & 9 & 5 \\ 3 & 0 & 1 & 4 \\ 0 & 10 & 2 & 1 \end{array}\right)$$

3 2 8 2

$$\left(\begin{array}{ccc|c} 2 & 5 & 9 & 5 \\ 0 & 9 & 4 & 2 \\ 0 & 10 & 2 & 1 \end{array}\right)$$

$9 \cdot 2 = 7$ $\quad 9 \cdot 4 = 3$ $\quad 9 \cdot 8 = 6$

$9 \cdot 5 = 1$ $\quad 9 \cdot 6 = 10$ $\quad 9 \cdot 9 = 4$

$9 \cdot 3 = 5$ $\quad 9 \cdot 7 = 8$ $\quad 9 \cdot 10 = 2$

0 10 2 1

$$\left(\begin{array}{ccc|c} 2 & 5 & 9 & 5 \\ 0 & 9 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$2x_1 + 5x_2 = 5$ $\quad 2 \cdot x_1 + 6 = 5$

$9x_2 = 2$ $\quad \underline{x_2 = 10}$ $\quad 2x_1 = 10$

$\underline{x_1 = 5}$

$2x_1 + 5x_2 + 9 = 0$

$9x_2 + 4 = 0$

$9x_2 = 7$ $\quad \underline{x_2 = 2}$

$2x_1 + 10 + 9 = 0$

$2x_1 = 3$ $\quad \Rightarrow \quad \underline{x_1 = 7}$

$x = \begin{pmatrix} 5 \\ 10 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$

164.

$$\mathfrak{A} = \begin{pmatrix} 2 & 5 & 8 & \cdots & 3n-1 \\ 5 & 8 & 11 & \cdots & 3n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n-1 & 3n+2 & 3n+5 & \cdots & 6n-4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 & 8 & \cdots & 3n-1 \\ 3 & 3 & 3 & & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 3 & 3 & \cdots & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 & 8 & \cdots & 3n-1 \\ 3 & 3 & 3 & \cdots & 3 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$n=1 \Rightarrow \operatorname{Rg} \mathfrak{A} = 1$

$n>1 \Rightarrow \operatorname{Rg} \mathfrak{A} = 2$

165. analog zu 164.

166.

$$\mathfrak{A} = \begin{pmatrix} -1 & 3 & 2 \\ -2 & 4 & 6 \\ 1 & -2 & 2 \end{pmatrix} \qquad \mathfrak{B} = \begin{pmatrix} -1 & 3 & 2 \\ 2 & -4 & 6 \\ 1 & -2 & 2 \end{pmatrix}$$

$$\mathfrak{A} \cdot \mathfrak{B} = \begin{pmatrix} 1+6+2 & -3-12-4 & -2+18+4 \\ 2+8+6 & -6-16-12 & -4+24+12 \\ -1-4+2 & 3+8-4 & 2-12+4 \end{pmatrix} = \begin{pmatrix} 9 & -19 & 20 \\ 16 & -34 & 32 \\ -3 & 7 & -6 \end{pmatrix}$$

$$\mathfrak{B} \cdot \mathfrak{A} = \begin{pmatrix} 1-6+2 & -3+12-4 & -2+18+4 \\ -2+8+6 & 6-16-12 & 4-24+12 \\ -1+4+2 & 3-8-4 & 2-12+4 \end{pmatrix} = \begin{pmatrix} -3 & 5 & 20 \\ 12 & -22 & -8 \\ 5 & -9 & -6 \end{pmatrix}$$

$$\mathfrak{A} \cdot \mathfrak{B} - \mathfrak{B} \cdot \mathfrak{A} = \begin{pmatrix} 12 & -24 & 0 \\ 4 & -12 & 40 \\ -8 & 16 & 0 \end{pmatrix}$$

167. $\cancel{\#}$ $a = \begin{pmatrix} -1 & 3 & 2 \\ -2 & 4 & 6 \\ 1 & -2 & 2 \end{pmatrix}$

$b = a^{-1}$

~~km~~ $\det a = -8 + 18 \cancel{+ 8} \cancel{- 8} \cancel{- 12} \cancel{+ 12} = 10$

$b_{11} = \frac{(-1)^{1+1}}{10} \cdot \begin{vmatrix} 4 & 6 \\ -2 & 2 \end{vmatrix} = \frac{1}{10} \cdot (8 + 12) = 2$

$b_{12} = \frac{(-1)^{1+1}}{10} \cdot \begin{vmatrix} 3 & 2 \\ -2 & 2 \end{vmatrix} = -\frac{1}{10} \cdot (6 + 4) = -1$

$b_{13} = \frac{(-1)^{1+3}}{10} \cdot \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = +\frac{1}{10} \cdot (18 - 8) = 1$

$b_{21} = \cancel{b} - \frac{1}{10} \cdot \begin{vmatrix} -2 & 6 \\ 1 & 2 \end{vmatrix} = -\frac{1}{10} \cdot (-4 - 6) = 1$

$b_{22} = \frac{1}{10} \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = \frac{1}{10} \cdot (-2 - 2) = -\frac{2}{5}$

$b_{23} = \frac{1}{10} \cdot \begin{vmatrix} -1 & 2 \\ -2 & 6 \end{vmatrix} = -\frac{1}{10} \cdot (-6 + 4) = \frac{1}{5}$

$b_{31} = \frac{1}{10} \cdot \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} = \frac{1}{10} \cdot (4 - 4) = 0$

$b_{32} = -\frac{1}{10} \cdot \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} = -\frac{1}{10} \cdot (2 - 3) = \frac{1}{10}$

$b_{33} = \frac{1}{10} \cdot \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} = \frac{1}{10} \cdot (-4 + 6) = \frac{1}{5}$

$$b = a^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & \frac{1}{10} & \frac{1}{5} \end{pmatrix}$$

167.

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ -2 & 4 & 6 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -2 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -2 & 2 & -2 & 1 & 0 \\ 0 & 0 & 5 & 0 & \frac{1}{2} & 1 \end{array}\right)$$

$$0 \quad 0 \quad 2 \quad 0 \quad \frac{1}{5} \quad \frac{2}{5}$$

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & +\frac{1}{5} & -\frac{2}{5} \\ 0 & -2 & 0 & -2 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 5 & 0 & \frac{1}{2} & 1 \end{array}\right) \quad 0 \quad -3 \quad 0 \quad -3 \quad \frac{6}{5} \quad -\frac{3}{5}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -2 & +1 & -1 \\ 0 & -2 & 0 & -2 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 5 & 0 & \frac{1}{2} & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{10} & \frac{1}{5} \end{array}\right)$$

168.

$$a = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ -1 & -2 & 2 \end{pmatrix} \quad b = \begin{pmatrix} -1 & 3 & 2 \\ 2 & -4 & 6 \\ 1 & -2 & 2 \end{pmatrix}$$

$$a \cdot b = \begin{pmatrix} -1+6+2 & 3-12-4 & 2+18+4 \\ -2+8+6 & 6-16-12 & 4+24+12 \\ +1-4+2 & -3+8-4 & -2-12+4 \end{pmatrix} = \begin{pmatrix} 7 & -13 & 24 \\ 12 & -22 & 40 \\ -1 & 1 & -10 \end{pmatrix}$$

$$b \cdot a = \begin{pmatrix} -1+6-2 & -3+12-4 & -2+18+4 \\ 2-8-6 & 6-16-12 & 4-24+12 \\ 1-4-2 & 3-8-4 & 2-12+4 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 20 \\ -12 & -22 & -8 \\ -5 & -9 & -6 \end{pmatrix}$$

$$a \cdot b - b \cdot a = \begin{pmatrix} 4 & -18 & 4 \\ 24 & 0 & 48 \\ 4 & 10 & -4 \end{pmatrix}$$

169.

$$a = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ -1 & -2 & 2 \end{pmatrix}$$

$$\det a = 8 - 18 \cancel{-8} \cancel{+8} \cancel{+12} \cancel{-12} = -10$$

$$a^{-1} = b$$

$$b_{11} = -\frac{1}{10} \cdot \begin{vmatrix} 4 & 6 \\ -2 & 2 \end{vmatrix} = -\frac{1}{10} \cdot (8+12) = -\frac{1}{10} \cdot 20 = -2$$

$$b_{12} = \frac{1}{10} \cdot \begin{vmatrix} 3 & 2 \\ -2 & 2 \end{vmatrix} = \frac{1}{10} \cdot (6+4) = 1$$

$$b_{13} = -\frac{1}{10} \cdot \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = -\frac{1}{10} \cdot (18-8) = -1$$

$$b_{21} = \frac{1}{10} \cdot \begin{vmatrix} 2 & 6 \\ -1 & 2 \end{vmatrix} = (4+6) \cdot \frac{1}{10} = 1$$

$$b_{22} = \frac{1}{10} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = -\frac{1}{10} \cdot (2+2) = -\frac{2}{5}$$

$$b_{23} = +\frac{1}{10} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = \frac{1}{10} \cdot (6-4) = \frac{1}{5}$$

$$b_{31} = -\frac{1}{10} \cdot \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} = -\frac{1}{10} \cdot (-4+4) = 0$$

$$b_{32} = \frac{1}{10} \cdot \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = \frac{1}{10} \cdot (-2+3) = \frac{1}{10}$$

$$b_{33} = -\frac{1}{10} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -\frac{1}{10} \cdot (4-6) = +\frac{1}{5}$$

$$a^{-1} = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & \frac{1}{10} & +\frac{1}{5} \end{pmatrix}$$

170. $\mathfrak{A} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$

$$\mathfrak{A} \cdot \mathfrak{A} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a \cdot c & 0 & 0 \end{pmatrix}$$

$$\mathfrak{A}^2 \cdot \mathfrak{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a \cdot c & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

171. wie 170.

172. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 1 & 7 & 2 & 5 & 4 & 3 & 6 \end{pmatrix} =$

$= (1\ 8\ 3)(2\ 9\ 6\ 5)(4\ 7) =$

$= (1\ 8)(1\ 3)(2\ 9)(2\ 6)(2\ 5)(4\ 7)$

$\operatorname{sgn} \pi = +1$

$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 8 & 7 & 6 & 9 & 4 & 1 & 2 \end{pmatrix}$

173. $k_k = (2, 3, \cancel{4}, 5, 4, 3, \overset{3}{\cancel{4}}, 2, 0, 0)$

1 2 3 4 5 6 7 8 9

8 9 7 5 6

~~8 4 9 1 7 2 3 5 4 6~~

~~8 9 7 5 6~~

8 9 1 7 2 5 4 3 6

174. analog zu 172.

175. $\pi = (1\ 3\ 8)(2\ 5\ 6\ 9)(4\ 7) =$

$= 47\ \cancel{1}2569\ 138$

176.

$$\det \mathcal{A} = \begin{vmatrix} -1 & 3 & 2 \\ -2 & 4 & 6 \\ 1 & -2 & 2 \end{vmatrix} = -8 + 18 \cancel{+8} \cancel{-8} - \cancel{12} + \cancel{12} = 10$$

$$\det \mathcal{B} = \begin{vmatrix} -1 & 3 & 2 \\ 2 & -4 & 6 \\ 1 & -2 & 2 \end{vmatrix} = 8 + 18 \cancel{-8} \cancel{+8} - 12 - 12 = 2$$

$$\begin{array}{r} 57 \cdot 32 \\ \hline 171. \\ 114 \\ \hline 1824 \end{array} \qquad \begin{array}{r} 34 \cdot 54 \\ \hline 170 \\ 136 \\ \hline 1836 \end{array} \qquad \begin{array}{r} 320 \cdot 7 \\ \hline 2240 \end{array}$$

$$\mathcal{A} \cdot \mathcal{B} = \begin{pmatrix} 9 & -19 & 20 \\ 16 & -34 & 32 \\ -3 & 7 & -6 \end{pmatrix}$$

$$\det \mathcal{A} \cdot \mathcal{B} = \begin{vmatrix} 9 & -19 & 20 \\ 16 & -34 & 32 \\ -3 & 7 & -6 \end{vmatrix} = 1836 + 1824 + 2240 -$$

$$-(2040 + 2016 + 1824) = 20$$

$$\begin{array}{r} 34 \cdot 60 \\ \hline 2040 \end{array} \qquad \begin{array}{r} 32 \cdot 63 \\ \hline 192. \\ 96 \\ \hline 2016 \end{array} \qquad \begin{array}{r} 19 \cdot 16 \\ 114 \\ \hline 304 \cdot 6 \\ \hline 1824 \end{array}$$

$$\begin{array}{r} 1836 \\ 1824 \\ 2240 \\ \hline 5900 \end{array} \qquad \begin{array}{r} 2040 \\ 2016 \\ 1824 \\ \hline 5880 \end{array}$$

$$\begin{array}{r} 5900 \\ -5880 \\ \hline 0020 \end{array}$$

$\det \mathcal{A} \cdot \det \mathcal{B} = \det(\mathcal{A} \cdot \mathcal{B})$

$10 \cdot 2 = 20$

177.

$$\det \mathcal{A} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ -1 & -2 & 2 \end{vmatrix} = 8 - 18 \cancel{- 8} \cancel{+ 8} \cancel{+ 12} \cancel{- 12} = -10$$

$$\det \mathcal{Z} = \begin{vmatrix} -1 & 3 & 2 \\ 2 & -4 & 6 \\ 1 & -2 & 2 \end{vmatrix} = 8 + 18 \cancel{- 8} \cancel{+ 8} \cancel{- 12} - 12 = 2$$

$$\det (\mathcal{A} \cdot \mathcal{Z}) = \begin{vmatrix} 4 & -18 & 4 \\ 24 & 0 & 48 \\ 4 & 10 & -4 \end{vmatrix} = 0 + -3456 + 960 - 0 - 1920 - 1728 = -7104 + 960 = -6144$$

72 · 48
288
576
3456

96 · 18
768
1728

3456
1920
1728
7104
+ 960
6144

$$\det (\mathcal{A} \cdot \mathcal{Z}) = \begin{vmatrix} 7 & -13 & 24 \\ 12 & -22 & 40 \\ -1 & 1 & -10 \end{vmatrix} = 1540 + 520 + 288 - 528 - 280 - 1560 =$$

22 · 70
1540

40 · 13
120
520

480
48
528

1440
120
1560

1540
520
288
2348

528
280
1560
2368

$$= 2348 - 2368 = -20$$

$$\det \mathcal{A} \cdot \det \mathcal{Z} = \det (\mathcal{A} \cdot \mathcal{Z})$$

$$-10 \cdot 2 = -20 \quad ✓$$

178. $\begin{vmatrix} 2 & 4 & -1 & 3 \\ 1 & 2 & 0 & -1 \\ 1 & 2 & 7 & 4 \\ 4 & 5 & 6 & 6 \end{vmatrix} =$

$= (-1)\cdot(-1)^{3+1}\cdot \begin{vmatrix} 1 & 2 & -1 \\ 1 & 2 & 4 \\ 4 & 5 & 6 \end{vmatrix} + 0 +$

$+ 7\cdot(-1)^{3+3}\cdot \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ 4 & 5 & 6 \end{vmatrix} +$

$+ 6\cdot(-1)^{3+4}\cdot \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ 1 & 2 & 4 \end{vmatrix} =$

$= -(\cancel{12} + 32 - 5 + 8 - 20 \cancel{-12}) +$

$+ 7\cdot(\cancel{24} - 16 + 15 \cancel{-24} + 10 - 24) + -$

$+ -6\cdot(\cancel{16} \cancel{-4} \cancel{+6} \cancel{-6} + 4 \cancel{-16}) =$

$= -15 - 7\cdot 15 = -8\cdot 15 = -120$

179. $\begin{vmatrix} 1 & 3 & -1 & 5 \\ 2 & 7 & 0 & 2 \\ -1 & -2 & 4 & 0 \\ 0 & 2 & 1 & -3 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 1 & -3 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & +1 & +13 \\ 0 & 0 & -3 & +13 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & +1 & +13 \\ 0 & 0 & 0 & 52 \end{vmatrix} = +52$

$$\begin{vmatrix} 1 & 3 & -1 & 5 \\ 2 & 7 & 0 & 2 \\ -1 & -2 & 4 & 0 \\ 0 & 2 & 1 & -3 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 1 & -3 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 1 & 13 \\ 0 & 0 & -3 & 13 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 52 \end{vmatrix} = 52$$

180. $\mathfrak{A} = \begin{pmatrix} x & 2 & 2 \\ 1 & 1 & x \\ 1 & x & -1 \end{pmatrix}$

$\det \mathfrak{A} = -x + 2x + 2x - 2 - x^3 + 2 =$

$= \quad 3x - x^3 = 0 \qquad x_1 = 0$

$x_{2,3} = \pm\sqrt{3} \notin \mathbb{Q}$

$\Rightarrow \underline{L = \{0\}}$

181. $\mathfrak{A} = \begin{pmatrix} 3 & x & 1 \\ 0 & 1 & x \\ x & -1 & 0 \end{pmatrix}$

$\det \mathfrak{A} = 0 + x^3 + 0 - x + 3x + 0 = x^3 + 2x$

$x_1 = 0$

$x^2 = -2 \qquad x = \pm\sqrt{-2} \notin \mathbb{Q}$

$\Rightarrow \underline{L = \{0\}}$

~~181a.~~ 182. $\det \mathfrak{A} = \begin{vmatrix} 6 & 3 & 7 \\ 8 & 5 & 9 \\ 9 & 3 & 10 \end{vmatrix} = 300 + 243 + 168 - 315 - 162 - 240 =$

$7 \cdot 3 = 21 \qquad \frac{21 \cdot 8}{168} \qquad \frac{35 \cdot 9}{315} \qquad = -15 + 6 + 3 = -6$

$9 \cdot 18 = 162$

$\Rightarrow \mathfrak{A}$ in $\mathbb{Z}_2$ und $\mathbb{Z}_3$ singulär.

183. $\det \mathfrak{A} = \begin{vmatrix} 2 & 2 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 + 0 + 0 - 0 - 2 - 16 = -14$

$\Rightarrow \mathfrak{A}$ in $\mathbb{Z}_2$ und $\mathbb{Z}_7$ singulär

184.

$$\det(\lambda \cdot E - A) = 0$$

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\det(\lambda \cdot E - A) = \det\begin{pmatrix} \lambda-3 & +1 \\ +1 & \lambda-3 \end{pmatrix} = (\lambda-3)^2 - 1 =$$

$$= \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_{1,2} = 3 \pm \sqrt{9-8}$$

$$\underline{\lambda_1 = 2}; \quad \underline{\lambda_2 = 4}$$

185. $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} \lambda-1 & +1 \\ +1 & \lambda-1 \end{pmatrix} = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1$$

$$\lambda^2 - 2\lambda = 0 \qquad \underline{\lambda_1 = 0}$$

$$\lambda - 2 = 0 \qquad \underline{\lambda_2 = 2}$$

186. $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \lambda^3 - \frac{1}{8} - \frac{1}{8} - \frac{1}{4}\lambda - \frac{1}{4}\lambda - \frac{1}{4}\lambda =$$

$$= \lambda^3 - \frac{1}{4} - \frac{3}{4}\lambda$$

$\lambda_1 = a$

$(\lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}) : (\lambda$