Prüfung Cenker 27.10.2004
(http://www.informatik-forum.at/showthread.php?t=23390)

1) Matrix A (1 2, 2 1), b = (-4,-2)
f: R^2 -> R mit f(x) = x^tAx+b^t*x

a) In welchen Punkten ist f stationär?
b) Besitzt f lokale/globale Extrema?
c) Bestimmung der Richtungsableitung im Punkt (1,1) in Richtung (4,-3)
d) In welche Richtung ist die Ableitung im Punkt (2,2) maximal?

2) g(x,y) = sin(xy)

a) Bestimmung der Tangentialabbildung im Punkt (Pi,1/2)
b) Implizite Form der Tangentialebene im Punkt (Pi,1/2) angeben
c) Bestimmung des Taylorpolynoms im Punkt (Pi, 1/2)
d) In welche Richtung ist der Anstieg von g(x,y) im Punkt (0,1) maximal?

3) B = {(x,y) € R^2: 4 <= x^2+y^2 <= 9, y >= 0}

a) Skizze des Bereichs B
b) Integral_B 1 dxdy
c) Integral_B xy dxdy

4) f(x,y) = y-x^2, g(x,y) = x^2+2y^2-1

a) Skizze der Niveaulinien von f
b) Bestimmung des Minimums von f(x,y) unter g(x,y) = 0
c) Minimierung von g(x,y) ausgehend vom Punkt (1,1) mit Hilfe der Methode
des steilsten Abstiegs

27.10.2004

1) Matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x) = x^t A x + b^t x$

a) f stationär

$$f(x,y) = (x,y)\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (-4,-2)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (x+2y\ ,\ 2x+y)\begin{pmatrix} x \\ y \end{pmatrix} -4x-2y$$

$$= x^2+2xy+2xy+y^2 -4x-2y$$

$$= x^2+y^2+4xy -4x-2y$$

$$\nabla f(x,y) = (2x+4y-4\ ,\ 2y+4x-2)$$

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$(2x+4y-4,\ 2y+4x-2) = (0,0)$$

$$\begin{aligned} 2x + 4y - 4 &= 0 \qquad {*2} \\ 4x + 2y - 2 &= 0 \end{aligned}$$

$$\begin{aligned} 2x+4y-4 &= 0 \\ 0 + 6y - 6 &= 0 \\ 6y &= 6 \\ y &= 1 \end{aligned}$$

$P_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

f ist im Punkt $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stationär

(1b)

$$\det\begin{pmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{pmatrix} = 4-4\lambda+\lambda^2 - 16 = \lambda^2-4\lambda-12=0$$

$$=(\lambda+2)\cdot(\lambda-6) = 0$$

$$\lambda_1 = -2$$

$$\lambda_2 = 6$$

Nicht positiv definit, daher kein lokales Maximum oder Minimum.
Daher auch kein globales Max oder Min

(1c)

$$x_0 = (1,1)$$

$$a = (4,-3)$$

$$a_n = \frac{a}{\|a\|} = \frac{(4,-3)}{\sqrt{4^2+(-3)^2}} = \frac{(4,-3)}{5} = \left(\frac{4}{5}, \frac{-3}{5}\right)$$

$$D_a f(x_0) = \nabla f(x_0)\cdot a = \langle \nabla f(x_0), a\rangle$$

$$= \left\langle (2x+4y-4, 4x+2y-2), \left(\frac{4}{5}, \frac{-3}{5}\right)\right\rangle$$

$$= \frac{4}{5}(2x+4y-4) + -\frac{3}{5}(4x+2y-2)$$

$$= \frac{8}{5}x + \frac{16}{5}y - \frac{16}{5} - \frac{12}{5}x - \frac{6}{5}y + \frac{6}{5} = -\frac{4}{5}x + \frac{10}{5}y - \frac{10}{5}$$

$$= -\frac{4}{5} + \frac{10}{5} - \frac{10}{5} = \underline{\underline{-\frac{4}{5}}}$$

(1d)

$$\vec{a} = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|} = \frac{(2,4)}{\|(2,4)\|} = \frac{(2,4)}{\sqrt{2^2+4^2}} = \frac{(2,4)}{\sqrt{20}} = \underline{\underline{\left(\frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}}\right)}}$$

② $g(x,y) = \sin(xy)$

a)

$$f(\pi,\tfrac{1}{2}) = g(\pi,\tfrac{1}{2}) + \nabla g(\pi,\tfrac{1}{2})\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix}$$

$$\nabla g = (y\cdot\cos(x\cdot y),\ x\cdot\cos(x\cdot y))$$

$\sin(\frac{\pi}{2}) = 1$

$\cos(\frac{\pi}{2}) = 0$

$$f(\pi,\tfrac{1}{2}) = \sin(\tfrac{\pi}{2}) + \left(\tfrac{1}{2}\cos(\tfrac{\pi}{2}),\ \pi\cdot\cos(\tfrac{\pi}{2})\right)\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix}$$

$$= \sin(\tfrac{\pi}{2}) + \tfrac{1}{2}\cos(\tfrac{\pi}{2})x - \tfrac{1}{2}\cos(\tfrac{\pi}{2})\pi + y\pi\cos(\tfrac{\pi}{2}) - \tfrac{\pi}{2}\cos(\tfrac{\pi}{2})$$

$$= 1 + 0 - 0 + 0 - 0 = 1$$

$$\nabla^2 g(x,y) = \begin{pmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy)\end{pmatrix}$$

b)

$z = 1$

$0 = 1 - z$

c)

$$g(x,y) = g(x_0,y_0) + \nabla g(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-x_0 \\ y-y_0\end{pmatrix}^t \nabla^2 g(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0\end{pmatrix} + \text{Rest}$$

$$\approx \sin(\tfrac{\pi}{2}) + \left(\tfrac{1}{2}\cos(\tfrac{\pi}{2}),\ \pi\cdot\cos(\tfrac{\pi}{2})\right)\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix} + \frac{1}{2}(x-\pi, y-\tfrac{1}{2})\,\nabla^2 g(\pi,\tfrac{1}{2})\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix}$$

$$\approx 1 + 0 + \frac{1}{2}(x-\pi, y-\tfrac{1}{2})\cdot\begin{pmatrix} -\frac{1}{4}\sin(\frac{\pi}{2}) & -\frac{\pi}{2}\sin(\frac{\pi}{2}) \\ -\frac{\pi}{2}\sin(\frac{\pi}{2}) & -\pi^2\sin(\frac{\pi}{2})\end{pmatrix}\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix}$$

$$\approx 1 + \frac{1}{2}\left(-\frac{1}{4}x + \frac{\pi}{4} - \frac{\pi}{2}y + \frac{\pi}{4},\ -\frac{\pi}{2}x + \frac{\pi^2}{2} - \pi^2 y + \frac{\pi^2}{2}\right)\begin{pmatrix} x-\pi \\ y-\frac{1}{2}\end{pmatrix}$$

$$\approx 1 + \frac{1}{2}\left(-\frac{1}{4}x^2 + \frac{1}{4}x\pi - \frac{\pi}{2}xy + \frac{\pi^2}{2}y + \frac{\pi}{2}x - \frac{\pi^2}{2} - \frac{\pi}{2}xy + \frac{\pi}{4}x + \frac{\pi^2}{2}y - \frac{\pi^2}{4} - \pi^2 y^2 - \frac{\pi^2}{2}y + \frac{\pi^2}{2}y - \frac{\pi^2}{4}\right)$$

$$\approx 1 + \frac{1}{2}\left(-\frac{x^2}{4} - \pi^2 y^2 - \pi xy + \pi x + \pi^2 y - \pi^2\right)$$

$$\approx -\frac{x^2}{8} - \frac{\pi^2 y^2}{2} - \frac{\pi xy}{2} + \frac{\pi x}{2} + \frac{\pi^2 y}{2} - \frac{\pi^2 + 1}{2}$$

2d) $a = \frac{\nabla g(x_0,y_0)}{\|\nabla g(x_0,y_0)\|} = \frac{(1\cdot\cos(0)\ ,\ 0)}{\|(1,0)\|} = (1,0)$

Der Anstieg im Punkt $(0,1)$ ist Richtung $(1,0)$ maximal

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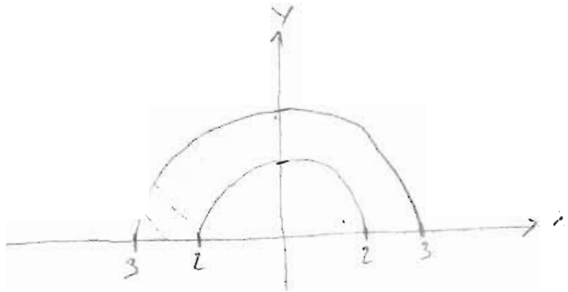
3) $B = \{(x,y)\in\mathbb{R}^2 : 4 \leq x^2+y^2 \leq 9\ , y \geq 0\}$

a) Skizze



b) $\int_B 1\,dx\,dy$ gesucht

$\Rightarrow$ Koordinatentransformation

$g(r,\varphi) \longmapsto (r\cos\varphi\ , r\sin\varphi)$ mit $x = r\cos\varphi$, $y = r\sin\varphi$

$g(r,\varphi) = (x,y)$, $r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$

$M = \{(r,\varphi) \mid 2 \leq r \leq 3,\ 0 \leq \varphi \leq \pi\}$

$\int_0^\pi \int_2^3 1\,r\,dr\,d\varphi$

$= \int^\pi \frac{r^2}{2}\Big|_2^3 d\varphi$

$\int_0^\pi \frac{9}{2} - \frac{4}{2}\ d\varphi = \int_0^\pi \frac{5}{2}\,d\varphi$

$= \frac{5}{2}\varphi\Big|_0^\pi = \underline{\underline{\frac{5}{2}\pi}}$

3c

$$\int_0^{\pi}\int_2^{3} r\cdot\cos\varphi \cdot r\cdot\sin\varphi \cdot r \, dr\, d\varphi$$

$$= \int_0^{\pi}\int_2^{3} r^3 \cos\varphi \sin\varphi \, dr\, d\varphi$$

$$= \int_0^{\pi} \frac{r^4}{4} \cos\varphi\cdot\sin\varphi \Big|_2^3 \, d\varphi$$

$$= \int_0^{\pi} \frac{81}{4}\cos\varphi\cdot\sin\varphi - \frac{16}{4}\cos\varphi \sin\varphi$$

$$= \int_0^{\pi} \frac{65}{4} \cos\varphi \sin\varphi$$

$$= \frac{65}{4}\cdot\frac{1}{2}\sin^2\varphi \Big|_0^{\pi}$$

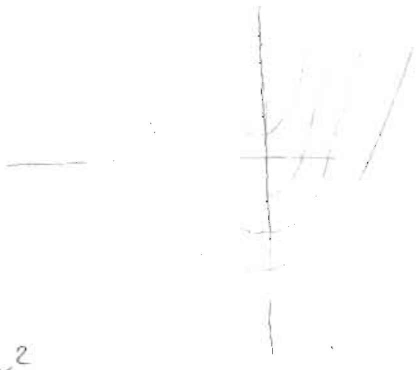
$$= \frac{65}{8}\sin^2\pi$$

$$= 0$$

4) $f(x,y) = y - x^2$

$g(x,y) = x^2 + 2y^2 - 1$

a) Niveaulinien von f



$z = y - x^2$

$y = z + x^2$

4b

$f(x,y) = y - x^2 \quad \to \text{MIN}$

$g(x,y) = x^2 + 2y^2 - 1 \quad = 0$

$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y))$

$L(x,y,\lambda) = y - x^2 - \lambda(x^2 + 2y^2 - 1)$

I: $\frac{\partial f}{\partial x} = -2x - 2x\lambda = 0 \qquad = -2x(1+\lambda) \Rightarrow x = 0, \; \lambda = -1$

II: $\frac{\partial f}{\partial y} = 1 - 4y\lambda = 0 \qquad \lambda = -1 : y = -\frac{1}{4}$

III: $\frac{\partial f}{\partial \lambda} = -x^2 - 2y^2 + 1 = 0$

$x = 0 \Rightarrow y^2 = \frac{1}{2} \qquad y = \pm\sqrt{\frac{1}{2}}$

$y = -\frac{1}{4} \Rightarrow x = \pm\sqrt{-2y^2 + 1} = \pm\sqrt{-2 \cdot \frac{1}{16} + 1} = \pm\sqrt{\frac{7}{8}}$

$P_1 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad P_3 = \begin{pmatrix} -\sqrt{\frac{7}{8}} \\ -\frac{1}{4} \end{pmatrix} \quad P_4 = \begin{pmatrix} \sqrt{\frac{7}{8}} \\ -\frac{1}{4} \end{pmatrix} \quad P_5 = \begin{pmatrix} \sqrt{\frac{7}{8}} \\ \frac{1}{4} \end{pmatrix} \quad P_6 = \begin{pmatrix} -\sqrt{\frac{7}{8}} \\ \frac{1}{4} \end{pmatrix}$

(4c) S38

stimmt wahrscheinlich nicht!

$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$g(x,y) = x^2 + 2y^2 - 1$

$(\nabla g(x,y))^t = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$

$$\varphi(\lambda) = g\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \end{pmatrix}\right) = (1-2\lambda)^2 + 2(1-4\lambda)^2 - 1$$

$$= 1 - 4\lambda + 4\lambda^2 + 2 - 16\lambda + 32\lambda^2 - 1$$

$$= 36\lambda^2 - 20\lambda + 2$$

$\varphi'(\lambda) = 72\lambda - 20$

$\lambda = \frac{20}{72} = \frac{5}{18}$

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{10}{18} \\ \frac{20}{18} \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix}$$

$$\varphi(\lambda) = g\left(\begin{pmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix} + \lambda \begin{pmatrix} \frac{8}{9} \\ -\frac{2}{9} \end{pmatrix}\right) = \left(\frac{4}{9} + \frac{8}{9}\lambda\right)^2 + 2\left(-\frac{1}{9} - \frac{2}{9}\lambda\right)^2 - 1 =$$

$$= \frac{16}{81} + \frac{64}{81}\lambda + \frac{64}{81}\lambda^2 + \frac{2}{81} + \frac{8}{81}\lambda + \frac{4}{81}\lambda^2 - \frac{81}{81}$$

$$= \frac{68}{81}\lambda^2 + \frac{72}{81}\lambda - \frac{63}{81}$$

$\varphi'(x) = \frac{136}{81}\lambda + \frac{72}{81}$

$\lambda = \frac{72}{136} = \frac{9}{17}$

$$x_2 = \begin{pmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix} + \lambda \begin{pmatrix} \frac{8}{9} \\ -\frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix} \cdot \begin{pmatrix} \frac{72}{153} \\ -\frac{18}{153} \end{pmatrix} = \begin{pmatrix} \frac{140}{153} \\ -\frac{35}{153} \end{pmatrix}$$

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